

# The Algebro-Geometric Solutions for the Ruijsenaars-Toda Hierarchy

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## Abstract

We provide a detailed treatment of Ruijsenaars-Toda (RT) hierarchy with special emphasis on its the theta function representation of all algebro-geometric solutions. The basic tools involve hyperelliptic curve  $\mathcal{K}_p$  associated with the Burchnell-Chaundy polynomial, Dubrovin-type equations for auxiliary divisors and associated trace formulas. With the help of a fundamental meromorphic function  $\phi$ , Baker-Akhiezer vector  $\Psi$  on  $\mathcal{K}_p$ , the complex-valued algebro-geometric solutions of RT hierarchy are derived.

## 1 Introduction

Nonlinear integrable lattice systems have been studied extensively in relation with various aspects and they usually possess rich mathematical structure such as Lax pairs, Hamilton structure, conservation law, etc. The Toda lattice is one of the most important integrable systems [7, 10]. It is well-known soliton equations such as the KdV, modified KdV, and nonlinear Schrödinger equations are closely related to or derived from the Toda equation by suitable limiting procedures [10, 12]. Various kinds of Toda lattice have been discussed since it was proposed [8, 9, 10, 11, 19]. Among them, a remarkable discovery was made by Ruijsenaars in the area of integrable lattice systems [1]. He found a relativistic integrable generalization of non-relativistic Toda lattice through solving a relativistic version of the Calogero-Moser system. The Lax representation, inverse scattering problem of the Ruijsenaars-Toda lattice and its connection with soliton dynamics

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were investigated. A general approach to construct relativistic generalizations of integrable lattice systems, applicable to the whole lattice KP hierarchy, was proposed by Gibbons and Kupershmidt [6]. After that, a series of techniques to construct relativistic lattice equations were developed by systematic procedure and Hirota's bilinear method [3, 17].

The Ruijsenaars-Toda lattice, sometimes also called relativistic Toda (RT) lattice, takes the form [3]

$$\begin{aligned}\beta_t &= (1 + \hbar\beta)(\alpha - \alpha^-), \\ \alpha_t &= \alpha(\beta^+ - \beta + \hbar\alpha^+ - \hbar\alpha^-)\end{aligned}\tag{1.1}$$

in Flaschka variables or

$$\begin{aligned}x_{k,tt} &= (1 + \hbar x_{k,t})(1 + \hbar x_{k+1,t}) \frac{e^{x_{k+1}-x_k}}{1 + \hbar^2 e^{x_{k+1}-x_k}} \\ &\quad - (1 + \hbar x_{k-1,t})(1 + \hbar x_{k,t}) \frac{e^{x_k-x_{k-1}}}{1 + \hbar^2 e^{x_k-x_{k-1}}}\end{aligned}\tag{1.2}$$

in Newtonian form, where the small time step  $\hbar = c^{-1}$  and  $c$  is light speed. In the non-relativistic limit  $c \rightarrow \infty$ , the RT equation (1.1) reduced to the well-known Toda lattice equation [7],

$$\beta_t = \alpha - \alpha^-, \quad \alpha_t = \alpha(\beta^+ - \beta)\tag{1.3}$$

in Flaschka variables, or

$$x_{k,tt} = e^{x_{k+1}-x_k} - e^{x_k-x_{k-1}}\tag{1.4}$$

in Newtonian form. Eq (1.1) is the Poincare-invariant generalizations of the Galilei-invariant Toda systems (1.3).

Mathematical frame work such as Lax representation, Bäcklund transformation, Hamiltonian structure of RT lattice Eq (1.1), etc, were investigated by some authors [13]-[16]. Cosentino obtained a soliton solution by using the IST method. Hietarinta and Junkichi Satsuma transformed the RT eq (1.1) into trilinear form through a suitable dependent variable transform. Later Yasuhiro Ohta, etc. decomposed the RT lattice eq (1.1) into three Toda systems, the Toda lattice itself, Bäcklund transformation of Toda lattice, and discrete time Toda lattice and explicitly derived the solutions in terms of the Casorati determinant[17]. The solution they obtained converges to that of TL eq (1.3) in the limit of  $c \rightarrow \infty$ .

Algebro-geometric solutions (finite-gap solutions or quasi-period solutions), as an important character of integrable system, is a kind of explicit

solutions closely related to the inverse spectral theory [27, 29]. Around 1975, several independent groups in UUSR and USA, namely, Novikov, Dubrovin and Krichever in Moscow, Matveev and Its in Leningrad, Lax, McKean, van Moerbeke and M. Kac in New York, and Marchenko, Kotlyarov and Kozel in Kharkov, developed the so-called finite-gap theory of nonlinear KdV equation based on the works of Drach, Burchnall and Chaunchy, and Baker [25, 28, 30]. The algebro-geometric method they established allowed us to find an important class of exact solutions to the soliton equations. As a degenerated case of this solutions, the multisoliton solutions and elliptic functions may be obtained [27, 35]. Its and Matveev first derived explicit expression of the quasi-period solution of KdV equation in 1975 [28], which is closely related to the finite-gap spectrum of the associated differential operator. Further exciting results appeared later, including the finite-gap solutions of Toda lattice, the Kadomtsev-Petviashvili equation and others [7, 30, 35], which could be found in the wonderful work of Belokolos, et al [27]. In recent years, a systematic approach based on the nonlinearization technique of Lax pairs or the restricted flow technique to derive the algebro-geometric solutions of  $(1+1)$ - and  $(2+1)$ -dimensional soliton equations has been obtained [31]-[34]. An alternate systematic approach proposed by Gesztesy and Holden can be used to construct algebro-geometric solutions has been extended to the whole  $(1+1)$  dimensional continuous and discrete hierarchy models [36]-[38],[40, 41].

In this paper, we mainly discussed the algebro-geometric quasi-period solutions of RT hierarchy for fixed constant  $\hbar = c^{-1} \neq 0$ . In following section 2, the RT equation (1.1) is extended to a whole RT hierarchy through the polynomial recursive relations. In section 3, we give a detailed study of algebro-geometric solutions for the stationary RT hierarchy. Firstly we derived the hyperelliptic curve  $\mathcal{K}_p$  in connection with the stationary RT hierarchy. Then a fundamental meromorphic function  $\phi$  on  $\mathcal{K}_p$ , the Baker-Akhiezer vector  $\Psi$  and the common eigenfunction of zero-curvature pair  $U, V_p$ , was introduced to study the trace formula and asymptotic properties of  $\phi$  and  $\psi_1$ , respectively. With the help of Riemann theta function associated with  $\mathcal{K}_p$ , one finds the theta function representations for  $\phi$  and  $\psi_1$  by alluding to Riemann's vanishing theorem and the Riemann-Roch theorem. In section 4, we derive the complex-valued algebro-geometric solutions of RT hierarchy with a given initial value problem by using the results in sections 3 and 4. Finally, in Appendix A and B we give Lagrange interpolation representation and asymptotic spectral parameter expansions that will be used in this paper.

## 2 The Ruijsenaars-Toda Hierarchy

In this section, we derive the Ruijsenaars-Toda hierarchy by using a polynomial recursion formalism. Throughout this section let us make the following assumption.

**Hypothesis 2.1.** In stationary case we assume that  $u : \mathbb{R} \rightarrow \mathbb{C}$  satisfies

$$\alpha, \beta \in \mathbb{C}^{\mathbb{Z}}, \quad \alpha(n) \neq 0, \quad n \in \mathbb{Z} \quad (2.1)$$

In the time-dependent case we suppose  $u : \mathbb{R}^2 \rightarrow \mathbb{C}$  satisfies

$$\begin{aligned} \alpha(\cdot, t), \beta(\cdot, t) &\in \mathbb{C}^{\mathbb{Z}}, \quad \alpha(n, t) \neq 0, \quad (n, t) \in \mathbb{Z} \times \mathbb{R} \\ \alpha(n, \cdot), \beta(n, \cdot) &\in C^1(\mathbb{R}). \end{aligned} \quad (2.2)$$

In this paper, we denote by  $S^{\pm}$  the shift operators acting on  $\psi = \{\psi(n)\}_{n=-\infty}^{+\infty} \in \mathbb{C}^{\mathbb{Z}}$  according to  $(S^{\pm}\psi)(n) = \psi(n \pm 1)$ , or  $\psi^{\pm} = S^{\pm}\psi$  for convenience.

We start from the following  $2 \times 2$  matrix isospectral problem [26]

$$S^+\psi = U\psi, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ (\hbar z - 1)\alpha & z + \beta \end{pmatrix}, \quad (2.3)$$

where the functions  $\alpha, \beta$  are potentials, and  $z$  is a constant spectral parameter independent of variable  $n$ .

Define sequences  $\{f_{\ell}\}_{\ell \in \mathbb{N}_0}$  and  $\{g_{\ell}\}_{\ell \in \mathbb{N}_0}$  recursively by

$$g_0 = -1/2, \quad f_0 = 0, \quad (2.4)$$

$$f_{\ell+1} + \beta f_{\ell} + g_{\ell} + g_{\ell}^{-} = 0, \quad \ell \in \mathbb{N}_0, \quad (2.5)$$

$$\begin{aligned} \hbar \alpha f_{\ell+1}^{-} - \hbar \alpha^{+} f_{\ell+1}^{+} - \alpha f_{\ell}^{-} + \alpha^{+} f_{\ell}^{+} - \beta g_{\ell}^{-} + \beta g_{\ell} \\ - g_{\ell+1}^{-} + g_{\ell+1} = 0, \quad \ell \in \mathbb{N}_0. \end{aligned} \quad (2.6)$$

Explicitly, one obtains

$$\begin{aligned} f_1 &= 1, \quad g_1 = \hbar \alpha^{+} - \delta_1/2, \\ f_2 &= -\hbar(\alpha^{+} + \alpha) - \beta + \delta_1, \\ g_2 &= -\hbar^2 \alpha^{++} \alpha^{+} - \hbar^2 (\alpha^{+})^2 + \hbar^2 \alpha^{+} \alpha - \hbar \alpha^{+} \beta^{+} - \hbar \alpha^{+} \beta - \alpha^{+} \\ &\quad + \hbar \alpha^{+} \delta_1 - \delta_2/2, \\ f_3 &= \hbar^2 \alpha^{++} \alpha^{+} + \hbar^2 (\alpha^{+})^2 - \hbar^2 \alpha^{+} \alpha + \hbar \alpha^{+} \beta^{+} + 2\hbar \alpha^{+} \beta + \alpha^{+} \\ &\quad + \hbar^2 \alpha^{+} \alpha + \hbar^2 \alpha^2 - \hbar^2 \alpha \alpha^{-} + 2\hbar \alpha \beta + \hbar \alpha \beta^{-} + \alpha - (\hbar \alpha^{+} \\ &\quad + \hbar \alpha + \beta) \delta_1 + \delta_2, \quad \text{etc.} \end{aligned} \quad (2.7)$$

Here  $\{\delta_\ell\}_{\ell \in \mathbb{N}}$  denote summation constants which naturally arise when solving (2.4). Subsequently, it will be useful to work with the corresponding homogeneous coefficients  $\hat{f}_\ell, \hat{g}_\ell$  defined by the vanishing of all summation constants  $\delta_k$  for  $k = 1, \dots, \ell$ ,

$$\begin{aligned}\hat{g}_0 &= -1/2, \quad \hat{g}_\ell = g_\ell|_{\delta_j=0, j=1, \dots, \ell}, \quad \ell \in \mathbb{N}, \\ \hat{f}_0 &= 0, \quad \hat{f}_\ell = f_\ell|_{\delta_j=0, j=1, \dots, \ell}, \quad \ell \in \mathbb{N}.\end{aligned}$$

By induction one infers that

$$\begin{aligned}g_\ell &= \sum_{s=0}^{\ell} \delta_{\ell-s} \hat{g}_s, \quad \ell \in \mathbb{N}_0 \\ f_\ell &= \sum_{s=0}^{\ell} \delta_{\ell-s} \hat{f}_s, \quad \ell \in \mathbb{N}_0,\end{aligned}\tag{2.8}$$

introducing

$$\delta_0 = 1.$$

**Remark 2.2.** The constants  $\delta_j$  ( $j \in \mathbb{N}_0$ ), can be expressed in terms of the branch points  $E_i$ , ( $i = 0, \dots, p+1$ ) of the associated spectral curve defined in (2.30). (Theorem 6)

In order to obtain the RT hierarchy associated with the spectral problem (2.3), we first solve the stationary zero-curvature equation

$$UV_p - V_p^+ U = 0, \quad V_p = (V_{ij})_{2 \times 2}\tag{2.9}$$

with

$$V_p = \begin{pmatrix} V_{11}^- & V_{12}^- \\ V_{21}^- & V_{22}^- \end{pmatrix},\tag{2.10}$$

where each entry  $V_{ij}$  is a polynomial in  $z$ ,

$$V_{11} = \sum_{j=0}^{p+1} g_{p+1-j} z^j + f_{p+2}, \quad V_{12} = \sum_{j=0}^{p+1} f_j z^{p+1-j},\tag{2.11}$$

$$V_{21} = (\hbar z - 1) \alpha^+ V_{12}^+ = (\hbar z - 1) \alpha^+ \left( \sum_{j=0}^{p+1} f_j^+ z^{p+1-j} \right),\tag{2.12}$$

$$V_{22} = - \sum_{j=0}^{p+1} g_{p+1-j} z^j.\tag{2.13}$$

Equation (2.9) can be rewritten as

$$V_{21}^- - (\hbar z - 1)\alpha V_{12} = 0, \quad (2.14)$$

$$V_{22}^- - V_{11} - (z + \beta)V_{12} = 0, \quad (2.15)$$

$$(\hbar z - 1)\alpha V_{11}^- + (z + \beta)V_{21}^- - (\hbar z - 1)\alpha V_{22} = 0, \quad (2.16)$$

$$(\hbar z - 1)\alpha V_{12}^- + (z + \beta)V_{22}^- - V_{21} - (z + \beta)V_{22} = 0. \quad (2.17)$$

Since  $\det(U) \neq 0$  for  $z \in \mathbb{C} \setminus \{1/\hbar\}$  by (2.1), (2.9) yields  $\text{tr}(V_p^+) = \text{tr}(UV_p U^{-1}) = \text{tr}(V_p)$  and hence

$$V_{11}^- + V_{22}^- = V_{11} + V_{22},$$

implying  $V_{11} + V_{22}$  is a lattice constant. If  $V_{11} + V_{22} = c \neq 0$ , one can add a polynomial times the identity to  $V_p$ , which does not affect the zero-curvature equation. Therefore we can choose

$$V_{11} + V_{22} = 0 \quad (2.18)$$

without loss of generality. This fact leads to the following result.

**Lemma 2.3.** Suppose  $U$  and  $V_p$  satisfy the stationary zero-curvature equation (2.9). Then (2.14)-(2.17) change into

$$V_{11}^- + V_{11} + (z + \beta)V_{12} = 0, \quad (2.19)$$

$$(\hbar z - 1)\alpha V_{12}^- - (z + \beta)V_{11}^- - (\hbar z - 1)\alpha^+ V_{12}^+ + (z + \beta)V_{11} = 0. \quad (2.20)$$

In particular, the coefficients  $\{f_\ell\}_{\ell=0,\dots,p}$  and  $\{g_\ell\}_{\ell=0,\dots,p}$  defined in (2.11)-(2.13) satisfy the recursive relations (2.4)-(2.6).

**Proof.** Eq.(2.19) and eq.(2.20) arise from (2.14)-(2.17) by substituting  $V_{22}$  for  $-V_{11}$ . Insertion of (2.11)-(2.13) into (2.19) and (2.20) then yields the relations (2.4)-(2.6).  $\square$

Inserting (2.11)-(2.13) into (2.9) and using the results of Lemma 2.3 yield the following theorem.

**Theorem 2.4.** Suppose that  $U$  and  $V_p$  satisfy the stationary zero-curvature equation (2.9). Then (2.9) reads

$$\begin{aligned} 0 &= UV_p - V_p^+ U \\ &= \begin{pmatrix} 0 & 0 \\ (\hbar z - 1)(\alpha f_{p+2}^- & -\alpha f_{p+1}^- + \alpha^+ f_{p+1}^+ \\ -\alpha f_{p+2}) & -\beta(g_{p+1}^- - g_{p+1}) \end{pmatrix}, \end{aligned} \quad (2.21)$$

which is equivalent to

$$f_{p+2}^- - f_{p+2} = 0, \quad (2.22)$$

$$\beta(g_{p+1}^- - g_{p+1}) + \alpha f_{p+1}^- - \alpha^+ f_{p+1}^+ = 0. \quad (2.23)$$

Thus, varying  $p \in \mathbb{N}_0$ , equations (2.22) and (2.23) give rise to the stationary Ruijsenaars-Toda (RT) hierarchy which we introduce as follows

$$\text{s-RT}_p(\alpha, \beta) = \begin{pmatrix} f_{p+2}^- - f_{p+2} \\ \beta(g_{p+1}^- - g_{p+1}) + \alpha f_{p+1}^- - \alpha^+ f_{p+1}^+ \end{pmatrix} = 0, \quad p \in \mathbb{N}_0. \quad (2.24)$$

In the special case  $p = 0$ , one obtains the stationary version of the Ruijsenaars-Toda system (1.1)

$$\begin{pmatrix} \hbar\alpha(\alpha^- - \alpha^+) + \alpha(\beta^- - \beta) \\ \hbar\beta(\alpha - \alpha^+) - \alpha + \alpha^+ \end{pmatrix} = 0. \quad (2.25)$$

In the case  $p = 1$ , one finds

$$\begin{pmatrix} \alpha(\hbar^2\alpha^{++}\alpha^+ + \hbar^2(\alpha^+)^2 - \hbar^2\alpha^+\alpha + \hbar\alpha^+\beta^+ + \alpha^+ \\ + \hbar\alpha\beta - \hbar\alpha\beta^- - \hbar^2\alpha\alpha^- - \hbar^2(\alpha^-)^2 + \hbar^2\alpha^-\alpha^{--} \\ - 2\hbar\alpha^-\beta^- - \hbar\alpha^-\beta^{--} - \alpha^-) \\ - \hbar\alpha^2 - \hbar\alpha\alpha^- - \alpha\beta^- + \hbar\alpha^+\alpha^{++} + \hbar(\alpha^+)^2 \\ + \alpha^+\beta^+ + \beta(-\hbar^2\alpha^+\alpha - \hbar^2\alpha^2 + \hbar^2\alpha\alpha^- \\ - \hbar\alpha\beta - \hbar\alpha\beta^- - \alpha + \hbar^2\alpha^{++}\alpha^+ + \hbar^2(\alpha^+)^2 \\ - \hbar^2\alpha^+\alpha + \hbar\alpha^+\beta^+ + \hbar\alpha^+\beta + \alpha^+) \end{pmatrix} \\ + \delta_1 \begin{pmatrix} \alpha(-\hbar\alpha^+ - \beta + \hbar\alpha^- + \beta^-) \\ \alpha - \alpha^+ + \hbar\alpha\beta - \hbar\alpha^+\beta \end{pmatrix} = 0.$$

In accordance with our notation introduced in (2.8), the corresponding homogeneous stationary Ruijsenaars-Toda equations are defined by

$$\widehat{\text{s-RT}}_p(\alpha, \beta) = \text{s-RT}_p(\alpha, \beta)|_{\delta_0=1, \delta_\ell=0, \ell=1, \dots, p}, \quad p \in \mathbb{N}_0.$$

Next we turn to the time-dependent Ruijsenaars-Toda hierarchy. For that purpose the potentials  $\alpha$  and  $\beta$  are now considered as functions of both the lattice point and time. For each equation in the hierarchy, that is, for each  $p \in \mathbb{N}_0$ , we introduce a deformation (time) parameter  $t_p \in \mathbb{R}$  in  $\alpha, \beta$ , replacing  $\alpha(n), \beta(n)$  by  $\alpha(n, t_p), \beta(n, t_p)$ . The quantities  $\{f_\ell\}_{\ell \in \mathbb{N}_0}$  and

$\{g_\ell\}_{\ell \in \mathbb{N}_0}$  are still defined by (2.4)-(2.6). The time-dependent Ruijsenaars-Toda hierarchy then obtained by imposing the zero-curvature equations

$$U_{t_p}(t_p) + U(t_p)V_p(t_p) - V_p^+(t_p)U(t_p) = 0, \quad t_p \in \mathbb{R}. \quad (2.26)$$

Relation (2.26) implies

$$\begin{pmatrix} 0 & 0 \\ (\hbar z - 1)\alpha_{t_p} & \beta_{t_p} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ (\hbar z - 1)(\alpha f_{p+2}^- & -\alpha f_{p+1}^- + \alpha^+ f_{p+1}^+ \\ -\alpha f_{p+2}^- & -\beta(g_{p+1}^- - g_{p+1}) \end{pmatrix} = 0. \quad (2.27)$$

Varying  $p \in \mathbb{N}_0$ , the collection of evolution equations

$$\text{RT}_p(\alpha, \beta) = \begin{pmatrix} \alpha_{t_p} - \alpha(f_{p+2} + f_{p+2}^-) \\ \beta_{t_p} - \beta(g_{p+1}^- - g_{p+1}) - \alpha f_{p+1}^- + \alpha^+ f_{p+1}^+ \end{pmatrix} = 0, \\ (n, t_p) \in \mathbb{Z} \times \mathbb{R}, \quad p \in \mathbb{N}_0, \quad (2.28)$$

then defines the time-dependent Ruijsenaars-Toda hierarchy. Explicitly,

$$\begin{aligned} \text{RT}_0(\alpha, \beta) &= \begin{pmatrix} \alpha_{t_0} - \hbar\alpha(\alpha^- - \alpha^+) - \alpha(\beta^- - \beta) \\ \beta_{t_0} - \hbar\beta(\alpha - \alpha^+) + \alpha - \alpha^+ \end{pmatrix} = 0, \\ \text{RT}_1(\alpha, \beta) &= \begin{pmatrix} \alpha_{t_1} - \alpha(\hbar^2\alpha^{++}\alpha^+ + \hbar^2(\alpha^+)^2 - \hbar^2\alpha^+\alpha \\ + \hbar\alpha^+\beta^+ + \alpha^+ + \hbar\alpha\beta - \hbar\alpha\beta^- - \hbar^2\alpha\alpha^- \\ - \hbar^2(\alpha^-)^2 + \hbar^2\alpha^-\alpha^{--} - 2\hbar\alpha^-\beta^- \\ - \hbar\alpha^-\beta^{--} - \alpha^-) \\ \beta_{t_1} + \hbar\alpha^2 + \hbar\alpha\alpha^- + \alpha\beta^- - \hbar\alpha^+\alpha^{++} - \hbar(\alpha^+)^2 \\ - \alpha^+\beta^+ - \beta(-\hbar^2\alpha^+\alpha - \hbar^2\alpha^2 + \hbar^2\alpha\alpha^- \\ - \hbar\alpha\beta - \hbar\alpha\beta^- - \alpha + \hbar^2\alpha^{++}\alpha^+ + \hbar^2(\alpha^+)^2 \\ - \hbar^2\alpha^+\alpha + \hbar\alpha^+\beta^+ + \hbar\alpha^+\beta + \alpha^+) \end{pmatrix} \\ &+ \delta_1 \begin{pmatrix} -\alpha(-\hbar\alpha^+ - \beta + \hbar\alpha^- + \beta^-) \\ -(\alpha - \alpha^+ + \hbar\alpha\beta - \hbar\alpha^+\beta) \end{pmatrix} = 0, \text{ etc.,} \end{aligned}$$

represent the first two equations of the time-dependent Ruijsenaars-Toda hierarchy. The system of equations,  $\text{RT}_0(\alpha, \beta) = 0$  is of course the Ruijsenaars-Toda system (1.1).

Next, taking into account (2.18), one infers that the expression  $R_{2p+2}$ , defined as

$$\begin{aligned} R_{2p+2}(z) &= -V_{11}^2(z, n) - V_{12}(z, n)V_{21}(z, n), \\ &= -V_{11}^2(z, n) - (\hbar z - 1)\alpha^+ V_{12}(z, n)V_{12}^+(z, n) \end{aligned} \quad (2.29)$$



is a lattice constant, that is,  $R_{2p+2} - R_{2p+2}^- = 0$ , since taking determinants in the stationary zero-curvature equation (2.9) immediately yields

$$(\hbar z - 1)\alpha \left( (-V_{11}^-)^2 - V_{12}^- V_{21}^- + V_{11}^2 + V_{12} V_{21} \right) = 0.$$

Hence,  $R_{2p+2}(z)$  only depends on  $z$ , and one may write  $R_{2p+2}$  as

$$R_{2p+2}(z) = -\frac{1}{4} \prod_{m=0}^{2p+1} (z - E_m), \quad \{E_m\}_{m=0}^{2p+1} \in \mathbb{C}, \quad p \in \mathbb{N}_0. \quad (2.30)$$

Relations (2.29) and (2.30) allows one to introduce a hyperelliptic curve  $\mathcal{K}_p$  of (arithmetic) genus  $p$  (possibly with a singular affine part), where

$$\mathcal{K}_p : \mathcal{F}_p(z, y) = y^2 + 4R_{2p+2} = y^2 - \prod_{m=0}^{2p+2} (z - E_m) = 0, \quad p \in \mathbb{N}_0. \quad (2.31)$$

Equations (2.19), (2.20) and (2.29) permit one to derive nonlinear difference equations for  $V_{11}, V_{12}$  separately. One obtains

$$\begin{aligned} & -V_{11}^2 - (\hbar z - 1)\alpha^+(V_{11}^+ + V_{11})(V_{11} + V_{11}^-) \\ & = (z + \beta)(z + \beta^+)R_{2p+2}, \end{aligned} \quad (2.32)$$

$$\begin{aligned} & -\left[ (\hbar z - 1)(\alpha V_{12}^- - \alpha^+ V_{12}^+) + (z + \beta)^2 V_{12} \right]^2 \\ & - 4(\hbar z - 1)(z + \beta)^2 \alpha^+ V_{12} V_{12}^+ = 4(z + \beta)^2 R_{2p+2} \end{aligned} \quad (2.33)$$

Equations analogous to (2.32) and (2.33) can be used to derive nonlinear recursion relations for homogenous coefficients  $\hat{f}_\ell, \hat{g}_\ell$ . In addition, as proven in Theorem B.1, (2.32) leads to an explicit determination of the summation constants  $\delta_1, \delta_2, \dots, \delta_p$  in (2.24) in terms of the zeros  $E_0, \dots, E_{2p+1}$  of associated polynomial  $R_{2p+2}$  in (2.30). In fact, one can prove (cf.(4.92))

$$\delta_\ell = c_\ell(\underline{E}), \quad \ell = 0, \dots, p, \quad (2.34)$$

where

$$\begin{aligned} c_0(\underline{E}) &= 1, \quad c_1(\underline{E}) = -\frac{1}{2} \sum_{m=0}^{2p+1} E_m \\ c_k(\underline{E}) &= \sum_{\substack{j_0, \dots, j_{2p+1}=0, \\ j_0 + \dots + j_{2p+1}=k}}^k \frac{(2j_0)! \dots (2j_{2p+1})! E_0^{j_0} \dots E_{2p+1}^{j_{2p+1}}}{2^{2k} (j_0!)^2 \dots (j_{2p+1}!)^2 (2j_0 - 1) \dots (2j_{2p+1} - 1)}, \\ & \quad k \in \mathbb{N}. \end{aligned}$$

are symmetric functions of  $\underline{E} = (E_0, \dots, E_{2p+1})$ .

**Remark 2.5.** If  $\alpha, \beta$  satisfy one of the stationary Ruijsenaars-Toda equations in (2.24) for a particular value of  $p$ ,  $\text{s-AL}_p(\alpha, \beta) = 0$ , then they satisfy infinitely many such equations of order higher than  $p$  for certain choices of summation constants  $\delta_\ell$ . This is seen as follows. Assume  $f_{p+2} - f_{p+2}^- = 0$  for some  $p \in \mathbb{N}$  and some set of integration constants  $\{\delta_\ell\}_{\ell=1, \dots, p} \subset \mathbb{C}$ , one infers

$$f_{p+2} = \lambda_{p+2}$$

for some constant  $\lambda_{p+2} \in \mathbb{C}$ . Subtracting the constant  $\lambda_{p+2}$  (i.e. writing  $f_{p+2} = \sum_{s=0}^{p+2} \check{\delta}_{p+2-k} \hat{f}_k$ , for some set of constants  $\{\check{\delta}_\ell\}_{\ell=1, \dots, p+2}$  and absorbing  $\lambda_{p+2}$  into  $\check{\delta}_{p+2}$ ), we may without loss of generality assume that  $f_{p+2} = 0$ , and hence the recursion (2.6) implies

$$\beta(g_{p+1}^- - g_{p+1}) + \alpha f_{p+1}^- - \alpha^+ f_{p+1}^+ = 0$$

in (2.24) is equivalent to

$$g_{p+2} - g_{p+2}^- - \hbar \alpha^+ f_{p+2}^- - \hbar \alpha f_{p+2}^- = 0.$$

Hence,

$$g_{p+2} = \lambda_{p+2}^*$$

for some constant  $\lambda_{p+2}^* \in \mathbb{C}$ . This indicates

$$\beta(g_{p+2}^- - g_{p+2}) + \alpha f_{p+2}^- - \alpha^+ f_{p+2}^+ = 0.$$

Then

$$f_{p+3} = \lambda_{p+3}$$

for some constant  $\lambda_{p+3} \in \mathbb{C}$  which arises from (2.5). Similarly, subtracting the constant  $\lambda_{p+3}$  (i.e. writing  $f_{p+3} = \sum_{s=0}^{p+3} \tilde{\delta}_{p+3-k} \hat{f}_k$ , for some set of constants  $\{\tilde{\delta}_\ell\}_{\ell=1, \dots, p+3}$  and absorbing  $\lambda_{p+3}$  into  $\tilde{\delta}_{p+3}$ ), we may without loss of generality assume that  $f_{p+3} = 0, \dots$ . Iterating this procedure yields

$$\text{s-RT}_q(\alpha, \beta) = 0$$

for all  $q \geq p+1$  (corresponding to some  $p$ -dependent choice of integration constants  $\{\check{\delta}_\ell\}_{\ell=1, \dots, p}$ ).

### 3 Stationary Algebro-geometric Solutions

This section is devoted to a detailed study of the stationary Ruijsenaars-Toda hierarchy and its algebro-geometric solutions. Our basic tools are derived from combining the polynomial recursion formalism introduced in Section 2 and a fundamental meromorphic function  $\phi$  on a hyperelliptic curve  $\mathcal{K}_p$ . We will obtain explicit Riemann theta function representations for the meromorphic function  $\phi$ , the Baker-Akhiezer function  $\psi_1$ , and the algebro-geometric solutions  $\alpha, \beta$ .

Unless explicitly stated otherwise, we suppose in this section that

$$\alpha, \beta \in \mathbb{C}^{\mathbb{Z}}, \quad \alpha(n) \neq 0, \quad n \in \mathbb{Z} \quad (3.1)$$

and assume (2.4)-(2.6), (2.9)-(2.11), (2.19)-(2.24), (2.29)-(2.31), keeping  $p \in \mathbb{N}_0$  fixed.

Throughout this section we assume  $\mathcal{K}_p$  defined in (2.31) to be nonsingular, that is, we suppose that

$$E_m \neq E_{m'} \text{ for } m \neq m', \quad E_m \in \mathbb{C} \setminus \{\hbar\}, \quad m = 0, \dots, 2p+1. \quad (3.2)$$

We compactify  $\mathcal{K}_p$  by adding two points  $P_{\infty+}$  and  $P_{\infty-}$ ,  $P_{\infty+} \neq P_{\infty-}$ , at infinity, still denoting its projective closure by  $\mathcal{K}_p$ . Finite points  $P$  on  $\mathcal{K}_p$  are denoted by  $P = (z, y)$  where  $y(P)$  denotes the meromorphic function on  $\mathcal{K}_p$  satisfying  $\mathcal{F}_p(z, y) = 0$ . The complex structure on  $\mathcal{K}_p$  is then defined in a standard manner and  $\mathcal{K}_p$  has topological genus  $p$ . Moreover, we use the involution

$$*: \mathcal{K}_p \rightarrow \mathcal{K}_p, \quad P = (z, y) \mapsto P^* = (z, -y), \quad P_{\infty\pm} \mapsto P_{\infty\pm}^* = P_{\infty\mp}. \quad (3.3)$$

We also emphasize that by fixing the curve  $\mathcal{K}_p$  (i.e., by fixing  $E_0, \dots, E_{2p+1}$ ), the summation constants  $\{\delta_\ell\}_{\ell=0, \dots, p}$  in the stationary  $\text{RT}_p$  equations are uniquely determined as is clear from (2.34), which establish the summation constants  $\delta_\ell$  as symmetric functions of  $E_0, \dots, E_{2p+1}$ .

For notational simplicity we will usually tacitly assume that  $p \in \mathbb{N}$ . (The trivial case  $p = 0$  is explicitly treated in Example 3.6)

In the following, the zeros of the polynomial  $V_{12}(\cdot, n)$  (cf. (2.11)) will play a special role. We denote them by  $\{\mu_j(n)\}_{j=1, \dots, p}$  and hence write

$$V_{12}(z) = \prod_{j=1}^p (z - \mu_j). \quad (3.4)$$

Similarly we write

$$V_{21}(z) = \hbar \alpha^+ (z - \frac{1}{\hbar}) \prod_{j=1}^p (z - \mu_j^+), \quad \mu_j^+(n) = \mu_j(n+1), \quad (3.5)$$

$$j = 1, \dots, p, \quad n \in \mathbb{Z},$$

and we recall that (cf.(2.29))

$$R_{2p+2} + V_{11}^2 = -(\hbar z - 1) \alpha V_{12} V_{12}^+. \quad (3.6)$$

The next step is crucial; it permits us to "lift" the zeros  $\mu_j$  from the complex plane  $\mathbb{C}$  to the curve  $\mathcal{K}_p$ . From (3.6) one infers that

$$R_{2p+2}(z) + V_{11}(z)^2 = 0, \quad z \in \{\mu_j, \mu_j^+, 1/\hbar\}_{j=1, \dots, p}.$$

Now we introduce  $\{\mu_j\}_{j=1, \dots, p} \subset \mathcal{K}_p$ ,  $\{\mu_j^+\}_{j=1, \dots, p} \subset \mathcal{K}_p$  and  $P_{\hbar} \in \mathcal{K}_p$  by

$$\hat{\mu}_j(n) = (\mu_j(n), -2V_{11}(\mu_j(n), n)), \quad j = 1, \dots, p \quad n \in \mathbb{Z}, \quad (3.7)$$

$$\hat{\mu}_j^+(n) = (\mu_j^+(n), 2V_{11}(\mu_j^+(n), n)), \quad j = 1, \dots, p \quad n \in \mathbb{Z}, \quad (3.8)$$

and

$$P_{\hbar} = (1/\hbar, 2V_{11}(1/\hbar, n)), \quad (3.9)$$

where

$$(2V_{11}(1/\hbar, n))^2 = y(1/\hbar)^2 = \prod_{m=0}^{2p+1} (\hbar - E_m)$$

is independent on  $n$ .

Next we briefly define some notations in connection with divisors on  $\mathcal{K}_p$  [37, 38]. A map,  $\mathcal{D} : \mathcal{K}_p \rightarrow \mathbb{Z}$ , is called a divisor on  $\mathcal{K}_p$  if  $\mathcal{D}(P) \neq 0$  for only finitely many  $P \in \mathcal{K}_p$ . The set of divisors on  $\mathcal{K}_p$  is denoted by  $\text{Div}(\mathcal{K}_p)$ . We shall employ the following (additive) notation for divisors,

$$\mathcal{D}_{Q_0 \underline{Q}} = \mathcal{D}_{Q_0} + \mathcal{D}_{\underline{Q}}, \quad \mathcal{D}_{\underline{Q}} = \mathcal{D}_{Q_1} + \dots + \mathcal{D}_{Q_m},$$

$$\underline{Q} = \{Q_1, \dots, Q_m\} \in \text{Sym}^m \mathcal{K}_p, \quad Q_0 \in \mathcal{K}_p, \quad m \in \mathbb{N},$$

where for any  $Q \in \mathcal{K}_p$ ,

$$\mathcal{D}_Q : \mathcal{K}_p \rightarrow \mathbb{N}_0, \quad P \mapsto \mathcal{D}_Q(P) = \begin{cases} 1 & \text{for } P = Q \\ 0 & \text{for } P \in \mathcal{K}_p \setminus \{Q\}, \end{cases}$$

and  $\text{Sym}^n \mathcal{K}_p$  denotes the  $n$ th symmetric product of  $\mathcal{K}_p$ . In particular, one can identify  $\text{Sym}^m \mathcal{K}_p$  with the set of nonnegative divisors  $0 \leq \mathcal{D} \in \text{Div}(\mathcal{K}_p)$

of degree  $m$ . Moreover, for a nonzero, meromorphic function  $f$  on  $\mathcal{K}_p$ , the divisor of  $f$  is denoted by  $(f)$ . Two divisors  $\mathcal{D}, \mathcal{E} \in \text{Div}(\mathcal{K}_p)$  are called equivalent, denoted by  $\mathcal{D} \sim \mathcal{E}$ , if and only if  $\mathcal{D} - \mathcal{E} = (f)$  for some  $f \in \mathcal{M}(\mathcal{K}_p) \setminus \{0\}$ . The divisor class  $[\mathcal{D}]$  of  $\mathcal{D}$  is then given by  $[\mathcal{D}] = \{\mathcal{E} \in \text{Div}(\mathcal{K}_p) | \mathcal{D} \sim \mathcal{E}\}$ . We recall that

$$\deg((f)) = 0, \quad f \in \mathcal{M}(\mathcal{K}_p) \setminus \{0\},$$

where the degree  $\deg(\mathcal{D})$  of  $\mathcal{D}$  is given by  $\deg(\mathcal{D}) = \sum_{P \in \mathcal{K}_p} \mathcal{D}(P)$ .

Next we introduce the fundamental meromorphic function on  $\mathcal{K}_p$  by

$$\phi(P, n) = \frac{y/2 - V_{11}(z, n)}{V_{12}(z, n)} \quad (3.10)$$

$$= \frac{(\hbar z - 1)\alpha^+ V_{12}^+(z, n)}{y/2 + V_{11}(z, n)}, \quad P = (z, y) \in \mathcal{K}_p, \quad n \in \mathbb{Z}, \quad (3.11)$$

with divisor  $(\phi(\cdot, n))$  of  $\phi(\cdot, n)$  given by

$$(\phi(\cdot, n)) = \mathcal{D}_{P_{\hbar\hat{\mu}^+(n)}} - \mathcal{D}_{P_{\infty - \hat{\mu}(n)}}, \quad (3.12)$$

using (3.4) and (3.5). Here we abbreviated

$$\hat{\mu} = \{\hat{\mu}_1, \dots, \hat{\mu}_p\}, \quad \hat{\mu}^+ = \{\hat{\mu}_1^+, \dots, \hat{\mu}_p^+\}.$$

Given  $\phi(\cdot, n)$ , the meromorphic stationary Baker-Akhiezer vector  $\Psi(\cdot, n, n_0)$  on  $\mathcal{K}_p$  is then defined by

$$\Psi(P, n, n_0) = \begin{pmatrix} \psi_1(P, n, n_0) \\ \psi_2(P, n, n_0) \end{pmatrix}, \quad (3.13)$$

$$\psi_1(P, n, n_0) = \begin{cases} \prod_{n'=n_0}^{n-1} \phi(P, n'), & n' > n_0, \\ 1, & n' = n_0, \\ \prod_{n'=n}^{n_0-1} \phi(P, n')^{-1}, & n' < n_0, \end{cases} \quad (3.14)$$

$$\psi_2(P, n, n_0) = \phi(P, n_0) \times \begin{cases} \prod_{n'=n_0+1}^{n-1} \left( \frac{\alpha(n')(\hbar z - 1)}{\phi^-(P, n')} + z + \beta(n') \right), & n' > n_0, \\ 1, & n' = n_0, \\ \prod_{n'=n+1}^{n_0} \left( \frac{\alpha(n')(\hbar z - 1)}{\phi^-(P, n')} + z + \beta(n') \right)^{-1}, & n' < n_0. \end{cases} \quad (3.15)$$

Basic properties of  $\phi$  and  $\Psi$  are summarized in the following result.

**Lemma 3.1.** Suppose  $\alpha, \beta$  satisfy (3.1) and the  $p$ th stationary Ruijsenaars-Toda system (2.24). Moreover, assume (2.30) (2.31) and (3.2) and let

$P = (z, y) \in \mathcal{K}_p \setminus \{P_{\infty\pm}, P_h\}$ ,  $(n, n_0) \in \mathbb{Z}^2$ . Then  $\phi$  satisfies the Riccati-type equation

$$\phi(P)\phi^-(P) - (z + \beta)\phi^-(P) - (\hbar z - 1)\alpha = 0, \quad (3.16)$$

as well as

$$\phi(P)\phi(P^*) = -\frac{(\hbar z - 1)\alpha^+ V_{12}^+(z)}{V_{12}(z)}, \quad (3.17)$$

$$\phi(P) + \phi(P^*) = -2\frac{V_{11}(z)}{V_{12}(z)}, \quad (3.18)$$

$$\phi(P) - \phi(P^*) = \frac{y}{V_{12}(z)}. \quad (3.19)$$

The vector  $\Psi$  satisfies

$$\psi_2(P, n, n_0) = \psi_1(P, n, n_0)\phi(P, n), \quad (3.20)$$

$$U(z)\Psi^-(P) = \Psi(P), \quad (3.21)$$

$$V_p(z)\Psi^-(P) = (y/2)\Psi^-(P), \quad (3.22)$$

$$\begin{aligned} \psi_1(P, n, n_0)\psi_1(P^*, n, n_0) &= (\hbar z - 1)^{n-n_0}\Gamma(\alpha^+, n, n_0) \\ &\quad \times \frac{V_{12}(z, n)}{V_{12}(z, n_0)}, \end{aligned} \quad (3.23)$$

$$\begin{aligned} \psi_1(P, n, n_0)\psi_2(P^*, n, n_0) + \psi_1(P^*, n, n_0)\psi_2(P, n, n_0) \\ = -2(\hbar z - 1)^{n-n_0}\Gamma(\alpha^+, n, n_0)\frac{V_{11}(z, n)}{V_{12}(z, n_0)}, \end{aligned} \quad (3.24)$$

$$\begin{aligned} \psi_1(P, n, n_0)\psi_2(P^*, n, n_0) - \psi_1(P^*, n, n_0)\psi_2(P, n, n_0) \\ = -(\hbar z - 1)^{n-n_0}\Gamma(\alpha^+, n, n_0)\frac{y}{V_{12}(z, n_0)}, \end{aligned} \quad (3.25)$$

where we used the abbreviation

$$\Gamma(f, n, n_0) = \begin{cases} \prod_{n'=n_0}^{n-1} f(n'), & n > n_0, \\ 1, & n = n_0, \\ \prod_{n'=n}^{n_0-1} f(n')^{-1}, & n < n_0. \end{cases} \quad (3.26)$$

**Proof.** To prove (3.16) one uses the definitions (3.10) of  $\phi$  and equations

(2.19) (2.20) and (2.29) to obtain

$$\begin{aligned}
& \phi(P)\phi^-(P) - (z + \beta)\phi^-(P) - (\hbar z - 1)\alpha \\
&= \frac{y/2 - V_{11}}{V_{12}} \frac{y/2 - V_{11}^-}{V_{12}^-} - (z + \beta) \frac{y/2 - V_{11}^-}{V_{12}^-} - (\hbar z - 1)\alpha \\
&= \frac{1}{V_{12}V_{12}^-} \left[ (y/2 - V_{11})(y/2 - V_{11}^-) - (z + \beta)V_{12}(y/2 - V_{11}^-) \right. \\
&\quad \left. - (\hbar z - 1)V_{12}V_{12}^- \right] \\
&= 0.
\end{aligned}$$

Equations (3.17)-(3.19) are clear from the definitions of  $\phi$  and  $y$ . Next we use induction to prove (3.20).

(i)  $n = n_0$ ; one easily finds

$$\psi_2(P, n_0, n_0) = \phi(P, n_0) = \psi_1(P, n_0, n_0)\phi(P, n_0).$$

by definition of  $\Psi$ .

(ii)  $n > n_0$ ; we assume (3.20) holds for  $n = n_0, \dots, n-1$ . Then  $\psi_1, \psi_2$  satisfy

$$\begin{aligned}
\frac{\psi_2(P, n, n_0)}{\psi_1(P, n, n_0)} &= \frac{(\hbar z - 1)\alpha\phi^-(P, n)^{-1} + z + \beta}{\phi^-(P, n)} \frac{\psi_2^-(P, n, n_0)}{\psi_1^-(P, n, n_0)} \\
&= \frac{(\hbar z - 1)\alpha\phi^-(P, n)^{-1} + z + \beta}{\phi^-(P, n)} \phi^-(P, n, n_0) \\
&= (\hbar z - 1)\alpha\phi^-(P, n)^{-1} + z + \beta,
\end{aligned} \tag{3.27}$$

that is,

$$\frac{\psi_2(P, n, n_0)}{\psi_1(P, n, n_0)} \phi^-(P, n) - (z + \beta)\phi^-(P, n) - (\hbar z - 1)\alpha = 0. \tag{3.28}$$

Comparing (3.28) with (3.16) then yields (3.20) for  $n \geq n_0, n \in \mathbb{N}$ .

(iii)  $n < n_0$ ; analogous proof with (ii).

The definition of  $\phi$  (cf. (3.14)) implies

$$\psi_1(P, n, n_0) = \psi_1^-(P, n, n_0)\phi^-(P, n) \tag{3.29}$$

and hence

$$\psi_1(P, n, n_0) = \psi_2^-(P, n, n_0), \tag{3.30}$$

which follows from (3.20) and (3.29). The definition of  $\psi_2$  (cf. (3.15)) implies

$$\begin{aligned}\psi_2(P, n, n_0) &= \left( \frac{(\hbar z - 1)\alpha}{\phi^-(P, n)} + z + \beta \right) \psi_2^-(P, n, n_0) \\ &= (\hbar z - 1)\alpha \psi_1^-(P, n, n_0) + (z + \beta)\psi_2^-(P, n, n_0),\end{aligned}\quad (3.31)$$

where we use (3.20) again. Then equation (3.21) follows from (3.30) and (3.31). Property (3.22) is an immediate consequence of (3.20) and the definition of  $\phi$ . Finally, Equations (3.23)-(3.25) follow from (3.17)-(3.19), the definition of  $\psi_1$  (cf. (3.14)) and (3.20).  $\square$

Combining the polynomial recursion approach in the section 2 with (3.4) yields the following trace formula, which means  $f_\ell, g_\ell$  can be expressed by the symmetric functions of the zeros  $\mu_j$  of  $V_{12}$ . For simplicity, we only show one of them.

**Lemma 3.2.** Suppose that  $\alpha, \beta$  satisfy the  $p$ th stationary Ruijsenaars-Toda system (2.24). Then,

$$-\hbar(\alpha + \alpha^+) - \beta + \delta_1 = -\sum_{j=1}^p \mu_j. \quad (3.32)$$

**Proof.** Relation (3.32) are proved by comparison of powers of  $z$  equating the corresponding expression (3.4) for  $V_{12}$  with that in (2.11) and with (2.7) taken into account.  $\square$

Next we turn to asymptotic properties of  $\phi$  and  $\psi_1$  in a neighborhood of  $P_{\infty\pm}$  and  $P_h$ . The asymptotic behavior of  $\Psi_2$  is derived naturally from (3.20). This is a crucial step to construct the stationary algebro-geometric solutions of stationary Ruijsenaars-Toda hierarchy.

**Lemma 3.3.** Suppose that  $\alpha, \beta$  satisfy the  $p$ -th stationary Ruijsenaars-Toda system (2.24). Moreover, let  $P = (z, y) \in \mathcal{K}_p \setminus \{P_{\infty\pm}, P_h\}$ ,  $(n, n_0) \in \mathbb{Z} \times \mathbb{Z}$ . Then,  $\phi$  has the asymptotic behavior

$$\phi(P) \underset{\zeta \rightarrow 0}{=} \begin{cases} \zeta^{-1} + (\beta - \hbar\alpha) + O(\zeta), & P \rightarrow P_{\infty-}, \\ \hbar\alpha^+ + (\hbar^2\alpha^+\alpha + \alpha - \hbar\alpha\beta)\zeta + O(\zeta^2), & P \rightarrow P_{\infty+}, \end{cases} \quad \zeta = 1/z, \quad (3.33)$$

$$\phi(P) \underset{\zeta \rightarrow 0}{=} \frac{\hbar\alpha^+}{\hbar + \beta^+}\zeta + O(\zeta^2), \quad P \rightarrow P_h, \quad \zeta = z - 1/\hbar, \quad (3.34)$$



Accordingly, the component  $\psi_1$  of the Baker-Akhiezer vector  $\Psi$  have the asymptotic behavior

$$\psi_1(P, n, n_0) \underset{\zeta \rightarrow 0}{=} \begin{cases} \zeta^{n_0-n} (1 + O(\zeta)), & P \rightarrow P_{\infty-}, \\ \Gamma(h\alpha^+) (1 + O(\zeta)), & P \rightarrow P_{\infty+}, \end{cases} \quad \zeta = 1/z, \quad (3.35)$$

$$\psi_1(P, n, n_0) \underset{\zeta \rightarrow 0}{=} \Gamma\left(\frac{h\alpha^+}{h + \beta^+}\right) \zeta^{n-n_0} (1 + O(\zeta)), \quad P \rightarrow P_h, \quad \zeta = z - 1/h. \quad (3.36)$$

The divisor  $(\psi_1)$  of  $\psi_1$  is given by

$$(\psi_1(\cdot, n, n_0)) = \mathcal{D}_{\hat{\mu}(n)} - \mathcal{D}_{\hat{\mu}(n_0)} + (n - n_0)(\mathcal{D}_{P_h} - \mathcal{D}_{P_{\infty-}}). \quad (3.37)$$

**Proof.** The existence of the asymptotic expansions of  $\phi$  in terms of the local coordinate  $\zeta = 1/z$  near  $P_{\infty\pm}$ , respectively,  $\zeta = z - 1/h$  near  $P_h$  is clear from the explicit form of  $\phi$  in (3.10) and (3.11). Insertion of the polynomials (2.11) and (2.12) into (3.10) and (3.11) then yields the explicit expansions coefficients in (3.33) and (3.34). Alternatively, and more efficiently, one can insert each of the following asymptotic expansions

$$\begin{aligned} \phi &\underset{\zeta \rightarrow 0}{=} \phi_{-1}\zeta^{-1} + \phi_0 + O(\zeta), \\ \phi &\underset{\zeta \rightarrow 0}{=} \phi_0 + \phi_1\zeta + O(\zeta^2), \\ \phi &\underset{\zeta \rightarrow 0}{=} \phi_1\zeta + \phi_2\zeta^2 + O(\zeta^3) \end{aligned} \quad (3.38)$$

into the Riccati-type equation (3.16) and, upon comparing coefficients of powers of  $\zeta$ , which determines the expansion coefficients  $\phi_k$  in (3.38), one concludes (3.33) and (3.34). Expansions (3.35) and (3.36) is an immediate consequence of (3.20), (3.33) and (3.34). Finally, expression (3.37) follows from (3.12) and (3.14).  $\square$

**Lemma 3.4.** Suppose that  $\alpha, \beta$  satisfy (3.1) and the  $p$ th stationary RT system (2.24). Moreover, assume hypothesis (2.31) and (3.2) and let  $n \in \mathbb{Z}$ . Let  $\mathcal{D}_{\hat{\mu}}$ ,  $\hat{\mu} = \{\hat{\mu}_1, \dots, \hat{\mu}_p\}$  be the Dirichlet divisor of degree  $p$  associated with  $\alpha, \beta$ , and  $\phi$  defined according to (3.10) and (3.11), that is

$$\hat{\mu}(n) = (\mu_j(n), -2V_{11}(\mu_j(n), n)) \in \mathcal{K}_p, \quad j = 1, \dots, p.$$

Then  $\mathcal{D}_{\hat{\mu}(n, t_r)}$  is nonspecial for all  $n \in \mathbb{Z}$ .

**Proof.** The divisor  $\mathcal{D}_{\hat{\mu}(n)}$  is nonspecial if and only if  $\{\hat{\mu}_1(n), \dots, \hat{\mu}_p(n)\}$  contains one pair of  $\{\hat{\mu}_j, \hat{\mu}_j^*(n)\}$ . Hence,  $\mathcal{D}_{\hat{\mu}(n)}$  is nonspecial as long as the projection  $\mu_j$  of  $\hat{\mu}_j$  are mutually distinct,  $\mu_j(n) \neq \mu_k(n)$  for  $j \neq k$ . If two or more projection coincide for some  $n_0 \in \mathbb{Z}$ , for instance,

$$\mu_{j_1}(n_0) = \dots = \mu_{j_k}(n_0) = \mu_0, \quad k > 1,$$

then there are two cases in the following associated with  $\mu_0$ .

(i)  $\mu_0 \notin \{E_0, E_1, \dots, E_{2p+1}\}$ ; we have  $V_{11}(\mu_0, n_0) \neq 0$  and  $\hat{\mu}_{j_1}(n_0), \dots, \hat{\mu}_{j_k}(n_0)$  all meet in the same sheet. Hence no special divisor can arise in this manner.

(ii)  $\mu_0 \in \{E_0, E_1, \dots, E_{2p+1}\}$ ; We assume  $\mu_0 = E_0$  without loss of generality. One concludes  $V_{12}(z, n_0) \underset{z \rightarrow E_0}{=} O((z - E_0)^2)$  and  $V_{11}(E_0, n_0) = 0$ . Hence

$$R_{2p+2}(z, n_0) = -V_{11}^2 - (\hbar z - 1)\alpha^+ V_{12} V_{12}^+ = O((\lambda - E_0)^2).$$

This conclusion contradict with the hypothesis (3.2) that the curve is non-singular. As a result, we have  $k = 1$  and  $\hat{\mu}_j$ ,  $j = 1, \dots, p$  are pairwise distinct. Then we have completed the proof.  $\square$

Next, we shall provide an explicit representation of  $\phi, \psi_1, \alpha$  and  $\beta$  in terms Riemann theta function associated with  $\mathcal{K}_p$ .

Let us introduce the holomorphic differentials  $\eta_\ell(P)$  on  $\mathcal{K}_p$  defined by

$$\eta_\ell = \frac{z^{\ell-1} dz}{y}, \quad \ell = 1, \dots, p$$

and choose an appropriate fixed homology basis  $\{a_j, b_j\}_{j=1}^{r-2}$  on  $\mathcal{K}_p$  in such a way that the intersection matrix of cycles satisfies

$$a_j \circ b_k = \delta_{j,k}, \quad a_j \circ a_k = 0, \quad b_j \circ b_k = 0, \quad j, k = 1, \dots, r-2.$$

Define an invertible matrix  $C \in GL(p, \mathbb{C})$  as follows

$$\begin{aligned} C &= (C_{j,k})_{p \times p}, \quad C_{j,k} = \int_{a_k} \eta_j, \\ \underline{c}(k) &= (c_1(k), \dots, c_p(k)), \quad c_j(k) = (C^{-1})_{j,k}, \end{aligned} \tag{3.39}$$

and the normalized holomorphic differentials

$$\omega_j = \sum_{\ell=1}^p c_j(\ell) \eta_\ell, \quad \int_{a_k} \omega_j = \delta_{j,k}, \quad \int_{b_k} \omega_j = \Gamma_{j,k}, \quad j, k = 1, \dots, p. \tag{3.40}$$

One can see that the matrix  $\Gamma = (\Gamma_{i,j})_{p \times p}$  is symmetric, and it has a positive-definite imaginary part.

Next, choosing a convenient base point  $Q_0 \in \mathcal{K}_p \setminus \{P_{\infty\pm}, P_h\}$ , the vector of Riemann constants  $\Xi_{Q_0}$  is given by (A.45) [37], and the Abel maps  $\underline{A}_{Q_0}(\cdot)$  and  $\underline{\alpha}_{Q_0}(\cdot)$  are defined by

$$\underline{A}_{Q_0} : \mathcal{K}_p \rightarrow J(\mathcal{K}_p) = \mathbb{C}^p / L_p,$$

$$\begin{aligned} P \mapsto \underline{A}_{Q_0}(P) &= (A_{Q_0,1}(P), \dots, A_{Q_0,p}(P)) \\ &= \left( \int_{Q_0}^P \omega_1, \dots, \int_{Q_0}^P \omega_p \right) \pmod{L_p}, \end{aligned}$$

and

$$\begin{aligned} \underline{\alpha}_{Q_0} : \text{Div}(\mathcal{K}_p) &\rightarrow J(\mathcal{K}_p), \\ \mathcal{D} \mapsto \underline{\alpha}_{Q_0}(\mathcal{D}) &= \sum_{P \in \mathcal{K}_p} \mathcal{D}(P) \underline{A}_{Q_0}(P), \end{aligned}$$

where  $L_p = \{\underline{z} \in \mathbb{C}^p \mid \underline{z} = \underline{N} + \Gamma \underline{M}, \underline{N}, \underline{M} \in \mathbb{Z}^p\}$ .

For brevity, define the function  $\underline{z} : \mathcal{K}_p \times \sigma^p \mathcal{K}_p \rightarrow \mathbb{C}^p$  by<sup>1</sup>

$$\begin{aligned} \underline{z}(P, \underline{Q}) &= \Xi_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}), \\ P \in \mathcal{K}_p, \underline{Q} &= (Q_1, \dots, Q_p) \in \sigma^p \mathcal{K}_p, \end{aligned} \tag{3.41}$$

here  $\underline{z}(\cdot, \underline{Q})$  is independent of the choice of base point  $Q_0$ . The Riemann theta function  $\theta(\underline{z})$  associated with  $\mathcal{K}_p$  and the homology is defined by

$$\theta(\underline{z}) = \sum_{\underline{n} \in \mathbb{Z}} \exp(2\pi i \langle \underline{n}, \underline{z} \rangle + \pi i \langle \underline{n}, \underline{n} \Gamma \rangle), \quad \underline{z} \in \mathbb{C}^p,$$

where  $\langle \underline{B}, \underline{C} \rangle = \overline{\underline{B}} \cdot \underline{C}^t = \sum_{j=1}^{r-2} \overline{B_j} C_j$  denotes the scalar product in  $\mathbb{C}^p$ .

Let  $\omega_{P_h P_{\infty-}}^{(3)}$  be the normal differential of the third kind holomorphic on  $\mathcal{K}_p \setminus \{P_{\infty+}, P_0\}$  with simple poles at  $P_{\infty-}$ ,  $P_h$  and residues -1, 1, respectively.

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<sup>1</sup>  $\sigma^p \mathcal{K}_p = \underbrace{\mathcal{K}_p \times \dots \times \mathcal{K}_p}_p$ .

In particular,

$$\omega_{P_h P_{\infty-}}^{(3)} = \frac{y - 2V_{11}(\hbar^{-1}, n)}{z - \hbar^{-1}} \frac{dz}{2y} + \frac{1}{2y} \prod_{j=1}^p (z - \lambda_j) dz \quad (3.42)$$

$$\stackrel{=}{\underset{\zeta \rightarrow 0}{}} \begin{cases} \left( -\zeta^{-1} + \left( -\frac{1}{4} \sum_{m=0}^{2p+1} E_m - \frac{1}{2} \hbar^{-1} + \frac{1}{2} \sum_{j=1}^p \lambda_j \right) + O(\zeta) \right) d\zeta, & P \rightarrow P_{\infty-}, \\ (\zeta^{-1} + O(1)) d\zeta, & P \rightarrow P_h, \end{cases} \quad (3.43)$$

where the constants  $\{\lambda_j\}_{j=1}^p \in \mathbb{C}$  are uniquely determined by employing the normalization

$$\int_{a_j} \omega_{P_h P_{\infty-}}^{(3)} = 0, \quad j = 1, \dots, p.$$

The explicit formula (3.42) and (3.43) then indicate the following asymptotic expansion near  $P_{\infty-}$  (using the local coordinate  $\zeta = 1/z$ ),

$$\begin{aligned} \exp \left( \int_{Q_0}^P \omega_{P_h P_{\infty-}}^{(3)} \right) &\stackrel{=}{\underset{\zeta \rightarrow 0}{}} c_0 \left( \zeta^{-1} + \left( -\frac{1}{4} \sum_{m=0}^{2p+1} E_m - \frac{1}{2} \hbar^{-1} + \frac{1}{2} \sum_{j=1}^p \lambda_j \right) + O(\zeta) \right), \quad P \rightarrow P_{\infty-} \end{aligned} \quad (3.44)$$

where  $c_0$  is an integration constant only depending on  $\mathcal{K}_p$ . Moreover, assume  $\eta \in \mathbb{C}$  and  $|\eta| < \min\{|E_0|^{-1}, |E_1|^{-1}, |E_2|^{-1}, \dots, |E_{2p+1}|^{-1}\}$  and abbreviate

$$\underline{E} = (E_0, E_1, \dots, E_{2p+1}).$$

Then

$$\left( \prod_{m=0}^{2p+1} (1 - E_m \eta) \right)^{-1/2} = \sum_{k=0}^{+\infty} \hat{c}_k(\underline{E}) \eta^k, \quad (3.45)$$

where

$$\hat{c}_0(\underline{E}) = 1, \quad \hat{c}_1(\underline{E}) = \frac{1}{2} \sum_{m=0}^{2p+1} E_m,$$

$$\hat{c}_k(\underline{E}) = \sum_{j_0, \dots, j_{2p+1}=0, j_0+\dots+j_{2p+1}=k}^k \frac{(2j_0)! \cdots (2j_{2p+1})!}{2^{2k} (j_0!)^2 (j_{2p+1}!)^2} E_0^{j_0} \cdots E_{2p+1}^{j_{2p+1}}, \quad k \in \mathbb{N}, \quad \text{etc.}$$

Similarly,

$$\left( \prod_{m=0}^{2p+1} (1 - E_m \eta) \right)^{1/2} = \sum_{k=0}^{+\infty} c_k(\underline{E}) \eta^k, \quad (3.46)$$

where

$$c_0(\underline{E}) = 1, \quad c_1(\underline{E}) = -\frac{1}{2} \sum_{m=0}^{2p+1} E_m,$$

$$c_k(\underline{E}) = \sum_{j_0, \dots, j_{2p+1}=0, j_0+\dots+j_{2p+1}=k}^k \frac{(2j_0)! \cdots (2j_{2p+1})! E_0^{j_0} \cdots E_{2p+1}^{j_{2p+1}}}{2^{2k} (j_0!)^2 \cdots (j_{2p+1}!)^2 (2j_0 - 1) \cdots (2j_{2p+1} - 1)} \\ k \in \mathbb{N}, \quad \text{etc.}$$

Given these preparations, the theta function representations of  $\phi, \psi_1, \alpha$ , and  $\beta$  then read as follows.

**Theorem 3.5.** Suppose that  $\alpha, \beta$  satisfy (3.1) and the  $p$ th stationary RT system (2.24). Moreover, assume hypothesis (2.31) and (3.2), and let  $P \in \mathcal{K}_p \setminus \{P_{\infty\pm}, P_h\}$  and  $(n, n_0) \in \mathbb{Z}^2$ . Then for each  $n \in \mathbb{Z}$ ,  $\mathcal{D}_{\hat{\mu}(n)}$  is nonspecial. Moreover,

$$\phi(P, n) = C(n) \frac{\theta(\underline{z}(P, \hat{\mu}^+(n)))}{\theta(\underline{z}(P, \hat{\mu}(n)))} \exp \left( \int_{Q_0}^P \omega_{P_h P_{\infty-}}^{(3)} \right), \quad (3.47)$$

$$\psi_1(P, n, n_0) = C(n, n_0) \frac{\theta(\underline{z}(P, \hat{\mu}(n)))}{\theta(\underline{z}(P, \hat{\mu}(n_0)))} \exp \left( (n - n_0) \int_{Q_0}^P \omega_{P_h P_{\infty-}}^{(3)} \right), \quad (3.48)$$

where

$$C(n) = c_0^{-1} \frac{\theta(\underline{z}(P_{\infty-}, \hat{\mu}(n)))}{\theta(\underline{z}(P_{\infty-}, \hat{\mu}^+(n)))} \quad (3.49)$$

and

$$C(n, n_0) = \begin{cases} \prod_{n'=n_0}^{n-1} C(n'), & n > n_0, \\ 1, & n = n_0, \\ \prod_{n'=n}^{n_0-1} C(n')^{-1}, & n < n_0. \end{cases} \quad (3.50)$$

The Abel map linearizes the auxiliary divisor  $\mathcal{D}_{\hat{\mu}(n)}$  in the sense that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}(n_0)}) - \underline{A}_{P_{\infty-}}(P_h)(n - n_0) \quad (3.51)$$

and  $\alpha, \beta$  are the form of

$$\alpha^+ = \frac{c_1}{\hbar c_0} \frac{\theta(\underline{z}(P_{\infty-}, \hat{\mu}(n)))}{\theta(\underline{z}(P_{\infty-}, \hat{\mu}^+(n)))} \frac{\theta(\underline{z}(P_{\infty+}, \hat{\mu}^+(n)))}{\theta(\underline{z}(P_{\infty+}, \hat{\mu}(n)))} \quad (3.52)$$

and

$$\begin{aligned} \beta &= (c_1/c_0) \frac{\theta(\underline{z}(P_{\infty-}, \hat{\mu}^-(n)))}{\theta(\underline{z}(P_{\infty-}, \hat{\mu}(n)))} \frac{\theta(\underline{z}(P_{\infty+}, \hat{\mu}(n)))}{\theta(\underline{z}(P_{\infty+}, \hat{\mu}^-(n)))} \\ &\quad - \frac{1}{4} \sum_{m=0}^{2p+1} E_m - \frac{1}{2} \hbar^{-1} + \frac{1}{2} \sum_{j=1}^p \lambda_j \\ &\quad + \frac{\partial}{\partial \omega_j} \ln \left( \frac{\theta(\underline{z}(P_{\infty-}, \hat{\mu}^+(n)) + \underline{\omega})}{\theta(\underline{z}(P_{\infty-}, \hat{\mu}(n)) + \underline{\omega})} \right) \Big|_{\underline{\omega}=0}. \end{aligned} \quad (3.53)$$

Here  $c_0, c_1 \in \mathbb{C}$  are integration constants.

**Proof.** By Lemma 3.4,  $\mathcal{D}_{\hat{\mu}(n)}$  is nonspecial and hence the theta functions defined in this lemma are not identical to zero. Obviously, by (3.12) and Riemann-Roch Theorem [21]

$$\phi(P, n) \frac{\theta(\underline{z}(P, \hat{\mu}(n)))}{\theta(\underline{z}(P, \hat{\mu}^+(n)))} \exp \left( - \int_{Q_0}^P \omega_{P_h P_{\infty-}}^{(3)} \right)$$

is holomorphic function on compact Riemann surface  $\mathcal{K}_p$  and therefore it is a constant  $C(n)$  related to  $n$ . Then  $\phi(P, n)$  has the form (3.47). Next we account for the following Taylor expansion near  $P_{\infty-}$  (with local coordinate  $\zeta = 1/z$ ),

$$\begin{aligned} \frac{\theta(\underline{z}(P, \hat{\mu}^+(n)))}{\theta(\underline{z}(P, \hat{\mu}(n)))} &\stackrel{\zeta \rightarrow 0}{=} \frac{\theta(\underline{z}(P_{\infty-}, \hat{\mu}^+(n)))}{\theta(\underline{z}(P_{\infty-}, \hat{\mu}(n)))} \left( 1 + \sum_{j=1}^p c_j(p) \right. \\ &\quad \times \frac{\partial}{\partial \omega_j} \ln \left( \frac{\theta(\underline{z}(P_{\infty-}, \hat{\mu}^+(n)) + \underline{\omega})}{\theta(\underline{z}(P_{\infty-}, \hat{\mu}(n)) + \underline{\omega})} \right) \Big|_{\underline{\omega}=0} \zeta \\ &\quad \left. + O(\zeta^2) \right), \quad \text{as } P \rightarrow P_{\infty-} \end{aligned} \quad (3.54)$$

and hence (3.44) and (3.54) indicate

$$\begin{aligned} \phi(P, n) &\underset{\zeta \rightarrow 0}{=} c_0 C(n) \frac{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}^+(n)))}{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}(n)))} \left( \zeta^{-1} + \left( -\frac{1}{4} \sum_{m=0}^{2p+1} E_m - \frac{1}{2} \hbar^{-1} \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \sum_{j=1}^p \lambda_j \right) - \frac{\partial}{\partial \omega_j} \ln \left( \frac{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}^+(n)) + \underline{\omega})}{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}(n)) + \underline{\omega})} \right) \Big|_{\underline{\omega}=0} + O(\zeta) \right), \\ &\quad \text{as } P \rightarrow P_{\infty-}. \end{aligned} \quad (3.55)$$

A comparison of the coefficients of the asymptotic relations (3.33) and (3.55) then yields the following expressions for  $C(n)$  and  $\beta - \hbar\alpha$ ,

$$C(n) = c_0^{-1} \frac{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}(n)))}{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}^+(n)))}, \quad (3.56)$$

$$\begin{aligned} \beta - \hbar\alpha &= -\frac{1}{4} \sum_{m=0}^{2p+1} E_m - \frac{1}{2} \hbar^{-1} + \frac{1}{2} \sum_{j=1}^p \lambda_j \\ &\quad + \frac{\partial}{\partial \omega_j} \ln \left( \frac{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}^+(n)) + \underline{\omega})}{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}(n)) + \underline{\omega})} \right) \Big|_{\underline{\omega}=0}. \end{aligned} \quad (3.57)$$

Similarly, one finds the following Taylor expansions near  $P_{\infty+}$  (with local coordinate  $\zeta = 1/z$ ),

$$\begin{aligned} \frac{\theta(\underline{z}(P, \hat{\underline{\mu}}^+(n)))}{\theta(\underline{z}(P, \hat{\underline{\mu}}(n)))} &\underset{\zeta \rightarrow 0}{=} \frac{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}^+(n)))}{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(n)))} \left( 1 - \sum_{j=1}^p c_j(p) \right. \\ &\quad \times \frac{\partial}{\partial \omega_j} \ln \left( \frac{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}^+(n)) + \underline{\omega})}{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(n)) + \underline{\omega})} \right) \Big|_{\underline{\omega}=0} \zeta \\ &\quad \left. + O(\zeta^2) \right), \quad \text{as } P \rightarrow P_{\infty+} \end{aligned} \quad (3.58)$$

and

$$\begin{aligned} \phi(P, n) &\underset{\zeta \rightarrow 0}{=} c_1 C(n) \frac{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}^+(n)))}{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(n)))} \left( 1 + c_2 \zeta + O(\zeta) \right), \\ &\quad \text{as } P \rightarrow P_{\infty+} \end{aligned} \quad (3.59)$$

where  $c_0, c_1$  are constants arising from the limiting procedure. A comparison of (3.33) and (3.59) then yields

$$\begin{aligned} \hbar\alpha^+ &= c_1 C(n) \frac{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}^+(n)))}{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(n)))} \\ &= c_1/c_0 \frac{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}(n)))}{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}^+(n)))} \frac{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}^+(n)))}{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(n)))}, \end{aligned} \quad (3.60)$$

which proves (3.52). By (3.57) and (3.60), one concludes (3.53). Finally, (3.51) is the immediate consequence of (3.12) and Abel's theorem [21, 22, 37, 38].  $\square$

We conclude this section with the trivial case  $p = 0$  excluded thus far.

**Example 3.6.** Assume  $p = 0$ ,  $P = (z, y) \in \mathcal{K}_0 \setminus \{P_{\infty\pm}, P_h\}$  and  $(n, n_0) \in \mathbb{Z}^2$ . Then,

$$V_0^+(z, n) = \begin{pmatrix} -z/2 - \hbar\alpha - \beta + \delta_1/2 & 1 \\ (\hbar z - 1)\alpha^+ & z/2 - \hbar\alpha^+ + \delta_1/2 \end{pmatrix},$$

$$\mathcal{K}_0 : \quad \mathcal{F}_0(z, y) = y^2 - (z - E_0)(z - E_1) = 0, \quad E_0, E_1 \in \mathbb{C}.$$

and  $\alpha, \beta$  satisfy

$$\hbar\alpha^+ + \hbar\alpha + \beta = -(E_0 + E_1)/2,$$

$$(\hbar\alpha^+ + (E_0 + E_1)/4)(-\hbar\alpha^+ + \delta_1/2) + \alpha^+ = -E_0 E_1/4,$$

that is,

$$\alpha = -\frac{E_0 + E_1}{2\hbar} \pm \frac{1}{2} \sqrt{\left(\frac{E_0 + E_1}{2\hbar}\right)^2 + \left(\frac{E_0 - E_1}{2}\right)^2}, \quad \beta = \frac{E_0 + E_1}{2}.$$

Moreover,

$$\begin{aligned} V_{12} &= 1, & V_{21} &= (\hbar z - 1)\alpha^+, \\ V_{11} &= -z/2 - \hbar\alpha - \beta + \delta_1/2, \\ V_{22} &= z/2 - \hbar\alpha^+ + \delta_1/2, \\ \phi(P, n_0) &= y/2 - (-z/2 - \hbar\alpha - \beta + \delta_1/2), \\ \Psi_1(P, n, n_0) &= \left(y/2 - (-z/2 - \hbar\alpha - \beta + \delta_1/2)\right)^{n-n_0}. \end{aligned}$$

## 4 Time-dependent Algebro-geometric Solutions

In this section we extend the algebro-geometric analysis of Section 3 to the time-dependent Ruijsenaars-Toda hierarchy.

For most of this section we assume the following hypothesis.



**Hypothesis 4.1.** (i) Suppose that  $\alpha, \beta$  satisfy

$$\begin{aligned} \alpha(\cdot, t), \beta(\cdot, t) &\in \mathbb{C}^{\mathbb{Z}}, \quad \alpha(n, t) \neq 0, \quad (n, t) \in \mathbb{Z} \times \mathbb{R}, \\ \alpha(n, \cdot), \beta(n, \cdot) &\in C^1(\mathbb{R}). \end{aligned} \quad (4.1)$$

(ii) Assume that the hyperelliptic curve  $\mathcal{K}_p$  satisfies (2.31) and (3.2).

The basic problem in the analysis of algebro-geometric solutions of the RT hierarchy consists in solving the time-dependent  $r$ th Ruijsenaars-Toda flow with initial data a stationary solution of the  $p$ th equation in the hierarchy. More precisely, given  $p \in \mathbb{N}_0$ , consider a solution  $\alpha^1(n), \beta^1(n)$  of the  $p$ th stationary Ruijsenaars-Toda system  $\text{s-RT}_p(\alpha^1, \beta^1) = 0$  associated with  $\mathcal{K}_p$  and a given set of integration constants  $\{\delta_\ell\}_{\ell=0}^p \subseteq \mathbb{C}$ . Next, let  $r \in \mathbb{N}_0$ ; we intend to construct solutions  $\alpha, \beta$  of the  $r$ th Ruijsenaars-Toda flow  $\text{RT}_r(\alpha, \beta) = 0$  with  $\alpha(n, t_{0,r}) = \alpha^1(n), \beta(n, t_{0,r}) = \beta^1(n)$  for some  $t_{0,r} \in \mathbb{R}$ . To emphasize that the integration constants in the definitions of the stationary and the time-dependent RT equations are independent of each other, we indicate this by adding a tilde on all the time-dependent quantities. Hence we shall employ the notation  $\tilde{V}_r, \tilde{\text{RT}}_r, \tilde{V}_{11}, \tilde{V}_{12}, \tilde{V}_{21}, \tilde{V}_{22}, \tilde{g}_\ell, \tilde{f}_\ell, \tilde{\delta}_\ell$ , in order to distinguish them from  $V_p, \text{RT}_p, V_{11}, V_{12}, V_{21}, V_{22}, g_\ell, f_\ell, \delta_\ell$ , in the following. In addition, we will follow a more elaborate notation inspired by Hirota's  $\tau$ -function approach and indicate the individual  $r$ th RLV flow by a separate time variable  $t_r \in \mathbb{R}$ .

The algebro-geometric initial value problem discussed above can be summed up in the form of zero-curvature equation

$$U_{t_r}(z, t_r) + U(z, t_r)\tilde{V}_r(z, t_r) - \tilde{V}_r^+(z, t_r)U(z, t_r) = 0, \quad (4.2)$$

$$U(z, t_r)V_p(z, t_r) - V_p^+(z, t_r)U(z, t_r) = 0. \quad (4.3)$$

For further reference, we recall the relevant quantities here (cf. (2.3), (2.10)-(2.13)):

$$U = \begin{pmatrix} 0 & 1 \\ (\hbar z - 1)\alpha & z + \beta \end{pmatrix}, \quad (4.4)$$

$$V_p = \begin{pmatrix} V_{11}^- & V_{12}^- \\ V_{21}^- & V_{22}^- \end{pmatrix}, \quad \tilde{V}_r = \begin{pmatrix} \tilde{V}_{11}^- & \tilde{V}_{12}^- \\ \tilde{V}_{21}^- & \tilde{V}_{22}^- \end{pmatrix}, \quad (4.5)$$

and

$$\begin{aligned}
V_{11} &= \sum_{j=0}^{p+1} g_{p+1-j} z^j + f_{p+2}, \quad V_{12} = \sum_{j=0}^{p+1} f_j z^{p+1-j} = \prod_{j=1}^p (z - \mu_j), \\
V_{21} &= (\hbar z - 1) \alpha^+ V_{12}^+ = (\hbar z - 1) \alpha^+ \left( \sum_{j=0}^{p+1} f_j^+ z^{p+1-j} \right), \\
V_{22} &= - \sum_{j=0}^{p+1} g_{p+1-j} z^j, \\
\tilde{V}_{11} &= \sum_{j=0}^{r+1} \tilde{g}_{r+1-j} z^j + \tilde{f}_{r+2}, \quad \tilde{V}_{12} = \sum_{j=0}^{r+1} \tilde{f}_j z^{r+1-j}, \\
\tilde{V}_{21} &= (\hbar z - 1) \alpha^+ \tilde{V}_{12}^+ = (\hbar z - 1) \alpha^+ \left( \sum_{j=0}^{r+1} \tilde{f}_j^+ z^{r+1-j} \right), \\
\tilde{V}_{22} &= - \sum_{j=0}^{r+1} \tilde{g}_{r+1-j} z^j
\end{aligned} \tag{4.6}$$

for fixed  $p \in \mathbb{N}_0 \setminus \{0\}, r \in \mathbb{N}_0$ . Here  $\{\tilde{g}_\ell\}_{\ell=0}^{r+1}, \{\tilde{f}_\ell\}_{\ell=0}^{r+2}$ , and  $\{g_\ell\}_{\ell=0}^{p+1}, \{f_\ell\}_{\ell=0}^{p+2}$  are defined by (2.4)-(2.6) corresponding to different constants  $\tilde{\delta}_\ell$  and  $\delta_\ell$ , respectively. Explicitly, (2.9) (cf. (2.19),(2.20)) and (2.26) are equivalent to

$$V_{11}^- + V_{11} + (z + \beta) V_{12} = 0, \tag{4.7}$$

$$(\hbar z - 1) \alpha V_{12}^- - (z + \beta) V_{11}^- - (\hbar z - 1) \alpha^+ V_{12}^+ + (z + \beta) V_{11} = 0, \tag{4.8}$$

$$\tilde{V}_{11} + (z + \beta) \tilde{V}_{12} - \tilde{V}_{22} = 0, \tag{4.9}$$

$$\alpha_{t_r} + \alpha \tilde{V}_{11}^- + (z + \beta) \alpha \tilde{V}_{12} - \alpha \tilde{V}_{22} = 0, \tag{4.10}$$

$$\beta_{t_r} + (\hbar z - 1) \alpha \tilde{V}_{12}^- + (z + \beta) \tilde{V}_{22} - (\hbar z - 1) \alpha^+ \tilde{V}_{12}^+ - (z + \beta) \tilde{V}_{11} = 0, \tag{4.11}$$

respectively. In particular, (2.29) holds in the present  $t_r$ -dependent setting, that is,

$$\begin{aligned}
R_{2p+2}(z) &= -V_{11}^2(z, n, t_r) - V_{12}(z, n, t_r) V_{21}(z, n, t_r), \\
&= -V_{11}^2(z, n, t_r) - (\hbar z - 1) \alpha^+ V_{12}(z, n, t_r) V_{12}^+(z, n, t_r).
\end{aligned} \tag{4.12}$$

Here we emphasize that  $R_{2p+2}$  is  $t_r$ -independence (cf. Lemma 4.3).

As in the stationary context, we introduce

$$\hat{\mu}_j(n, t_r) = (\mu_j(n, t_r), -2V_{11}(\mu_j(n, t_r), n, t_r)), \quad j = 1, \dots, p \quad n \in \mathbb{Z}, \quad (4.13)$$

$$\hat{\mu}_j^+(n, t_r) = (\mu_j^+(n, t_r), 2V_{11}(\mu_j^+(n, t_r), n, t_r)), \quad j = 1, \dots, p \quad n \in \mathbb{Z}, \quad (4.14)$$

and note that the regularity assumptions (4.1) on  $\alpha, \beta$  imply continuity of  $\mu_j$  with respect to  $t_r \in \mathbb{R}$ .

In analogy to (3.10), (3.11), one defines the following meromorphic function  $\phi(\cdot, n, t_r)$  on  $\mathcal{K}_p$ ,

$$\phi(P, n) = \frac{y/2 - V_{11}(z, n, t_r)}{V_{12}(z, n, t_r)} \quad (4.15)$$

$$= \frac{(\hbar z - 1)\alpha^+(n, t_r)V_{12}^+(z, n, t_r)}{y/2 + V_{11}(z, n, t_r)}, \quad (4.16)$$

$$P = (z, y) \in \mathcal{K}_p, \quad (n, t_r) \in \mathbb{Z} \times \mathbb{R},$$

with divisor  $(\phi(\cdot, n, t_r))$  of  $\phi(\cdot, n, t_r)$  given by

$$(\phi(\cdot, n, t_r)) = \mathcal{D}_{P_0 \hat{\mu}^+(n, t_r)} - \mathcal{D}_{P_\infty - \hat{\mu}(n, t_r)}. \quad (4.17)$$

The time-dependent Baker-Akhiezer vector is then defined in terms of  $\phi$  by

$$\Psi(P, n, n_0, t_r, t_{0,r}) = \begin{pmatrix} \psi_1(P, n, n_0, t_r, t_{0,r}) \\ \psi_2(P, n, n_0, t_r, t_{0,r}) \end{pmatrix}, \quad (4.18)$$

$$\begin{aligned} \psi_1(P, n, n_0, t_r, t_{0,r}) &= \exp \left( \int_{t_{0,r}}^{t_r} \left( \tilde{V}_{11}(z, n_0, s) + \tilde{V}_{12}(z, n_0, s) \phi(P, n_0, s) \right) ds \right) \\ &\times \begin{cases} \prod_{n'=n_0}^{n-1} \phi(P, n', t_r), & n' > n_0, \\ 1, & n' = n_0, \\ \prod_{n'=n}^{n_0-1} \phi(P, n', t_r)^{-1}, & n' < n_0, \end{cases} \end{aligned} \quad (4.19)$$

$$\begin{aligned} \psi_2(P, n, n_0) &= \exp \left( \int_{t_{0,r}}^{t_r} \left( \tilde{V}_{11}(z, n_0, s) + \tilde{V}_{12}(z, n_0, s) \phi(P, n_0, s) \right) ds \right) \\ &\times \phi(P, n_0, t_r) \times \begin{cases} \prod_{n'=n_0+1}^{n-1} \left( \frac{\alpha(n', t_r)(\hbar z - 1)}{\phi^-(P, n', t_r)} + z + \beta(n', t_r) \right), & n' > n_0, \\ 1, & n' = n_0, \\ \prod_{n'=n+1}^{n_0} \left( \frac{\alpha(n', t_r)(\hbar z - 1)}{\phi^-(P, n', t_r)} + z + \beta(n', t_r) \right)^{-1}, & n' < n_0, \end{cases} \end{aligned} \quad (4.20)$$

$$P = (z, y) \in \mathcal{K}_p \setminus \{P_{\infty \pm}, P_0\}, \quad (n, t_r) \in \mathbb{Z} \times \mathbb{R}. \quad (4.21)$$

One observes that

$$\begin{aligned} \psi_1(P, n, n_0, t_r, t_{0,r}) &= \psi_1(P, n_0, n_0, t_r, t_{0,r}) \psi_1(P, n, n_0, t_r, t_r) \\ P &\in \mathcal{K}_p \setminus \{P_{\infty\pm}, P_0\}, (n, n_0, t_r, t_{0,r}) \in \mathbb{Z}^2 \times \mathbb{R}^2. \end{aligned} \quad (4.22)$$

The following lemma records basic properties of  $\phi$  and  $\Psi$  in analogy to the stationary case discussed in Lemma 3.1.

**Lemma 4.2.** Assume Hypothesis 4.1 and suppose that (4.2), (4.3) hold. In addition, let  $P = (z, y) \in \mathcal{K}_p \setminus \{P_{\infty\pm}, P_h\}, (n, n_0, t_r, t_{0,r}) \in \mathbb{Z}^2 \times \mathbb{R}^2$ . Then  $\phi$  satisfies

$$\phi(P)\phi^-(P) - (z + \beta)\phi^-(P) - (\hbar z - 1)\alpha = 0, \quad (4.23)$$

$$\phi_{t_r}(P) = (-\tilde{V}_{11} + \tilde{V}_{22})\phi - \tilde{V}_{12}\phi^2 + (\hbar z - 1)\alpha^+\tilde{V}_{12}^+, \quad (4.24)$$

$$\phi(P)\phi(P^*) = -\frac{(\hbar z - 1)\alpha^+V_{12}^+(z)}{V_{12}(z)}, \quad (4.25)$$

$$\phi(P) + \phi(P^*) = -2\frac{V_{11}(z)}{V_{12}(z)}, \quad (4.26)$$

$$\phi(P) - \phi(P^*) = \frac{y}{V_{12}(z)}. \quad (4.27)$$

Moreover, assuming  $P = (z, y) \setminus \{P_{\infty\pm}, P_h\}$ , then  $\Psi$  satisfies

$$\psi_2(P, n, n_0, t_r, t_{0,r}) = \psi_1(P, n, n_0, t_r, t_{0,r})\phi(P, n, t_r), \quad (4.28)$$

$$U(z)\Psi^-(P) = \Psi(P), \quad (4.29)$$

$$V_p(z)\Psi^-(P) = (y/2)\Psi^-(P), \quad (4.30)$$

$$\Psi_{t_r}(p) = \tilde{V}_r^+(z)\Psi(P), \quad (4.31)$$

$$\begin{aligned} \psi_1(P, n, n_0, t_r, t_{0,r})\psi_1(P^*, n, n_0, t_r, t_{0,r}) &= (\hbar z - 1)^{n-n_0}\Gamma(\alpha^+, n, n_0, t_r) \\ &\times \frac{V_{12}(z, n, t_r)}{V_{12}(z, n_0, t_{0,r})}, \end{aligned} \quad (4.32)$$

$$\begin{aligned} \psi_1(P, n, n_0, t_r, t_{0,r})\psi_2(P^*, n, n_0, t_r, t_{0,r}) &+ \psi_1(P^*, n, n_0, t_r, t_{0,r})\psi_2(P, n, n_0, t_r, t_{0,r}) \\ &= -2(\hbar z - 1)^{n-n_0}\Gamma(\alpha^+, n, n_0, t_r)\frac{V_{11}(z, n, t_r)}{V_{12}(z, n_0, t_{0,r})}, \end{aligned} \quad (4.33)$$

$$\begin{aligned} \psi_1(P, n, n_0, t_r, t_{0,r})\psi_2(P^*, n, n_0, t_r, t_{0,r}) &- \psi_1(P^*, n, n_0, t_r, t_{0,r})\psi_2(P, n, n_0, t_r, t_{0,r}) \\ &= -(\hbar z - 1)^{n-n_0}\Gamma(\alpha^+, n, n_0, t_r)\frac{y}{V_{12}(z, n_0, t_{0,r})}, \end{aligned} \quad (4.34)$$

where we used the abbreviation

$$\Gamma(f, n, n_0, t_r) = \begin{cases} \prod_{n'=n_0}^{n-1} f(n', t_r), & n > n_0, \\ 1, & n = n_0, \\ \prod_{n'=n}^{n_0-1} f(n', t_r)^{-1}, & n < n_0. \end{cases} \quad (4.35)$$

Moreover, as long as the zeros of  $\mu_j(n_0, s)$  of  $V_{12}(\cdot, n_0, s)$  are all simple for all  $s \in \Omega$ ,  $\Omega \subseteq \mathbb{R}$ , is an open interval,  $\Psi$  is meromorphic on  $\mathcal{K}_p \setminus \{P_{\infty\pm}, P_h\}$  for  $(n, n_0, t_r) \in \mathbb{Z} \times \Omega^2$ .

**Proof.** Equations (4.23), (4.25)-(4.30), (4.32)-(4.34) are proved as in the stationary case, see Lemma 3.1. Thus, we turn to the proof of (4.24) and (4.31): Differentiating the Riccati-type equation (4.23) yields

$$\phi_{t_r} \phi^- + \phi \phi_{t_r}^- = \beta_{t_r} \phi^- + (z + \beta) \phi_{t_r}^- + (\hbar z - 1) \alpha_{t_r},$$

that is,

$$\begin{aligned} & -(\phi^- + (\phi - (z + \beta)) S^-) \phi_{t_r} \\ & = \left( (\hbar z - 1) \alpha \tilde{V}_{12}^- + (z + \beta) \tilde{V}_{22}^- - (\hbar z - 1) \alpha^+ \tilde{V}_{12}^+ - (z + \beta) \tilde{V}_{22}^+ \right) \phi^- \\ & + (\hbar z - 1) \left( \alpha \tilde{V}_{11}^- + (z + \beta) \alpha \tilde{V}_{12} - \alpha \tilde{V}_{22} \right) \end{aligned}$$

by using (4.10) and (4.11). This allows one to calculate the righthand side of (4.24)

$$\begin{aligned} & (\phi^- + (\phi - (z + \beta)) S^-) \left( \tilde{V}_{11} \phi + \tilde{V}_{12} \phi^2 - \tilde{V}_{21} - \tilde{V}_{22} \phi \right) \\ & = \tilde{V}_{11} \phi \phi^- + \tilde{V}_{12} \phi^2 \phi^- - \tilde{V}_{21} \phi^- - \tilde{V}_{22} \phi \phi^- + (\hbar z - 1) \alpha \tilde{V}_{11}^- \\ & + (\hbar z - 1) \alpha \tilde{V}_{12}^- \phi^- - \tilde{V}_{21}^- \phi + (z + \beta) \tilde{V}_{21}^- - \tilde{V}_{22}^- (\hbar z - 1) \alpha \\ & = \tilde{V}_{11} \left( (\hbar z - 1) \alpha + (z + \beta) \phi^- \right) + \tilde{V}_{12} \phi \left( (\hbar z - 1) \alpha + (z + \beta) \phi^- \right) \\ & - \tilde{V}_{21} \phi^- - \tilde{V}_{22} \left( (\hbar z - 1) \alpha + (z + \beta) \phi^- \right) + (\hbar z - 1) \alpha \tilde{V}_{11}^- + (\hbar z - 1) \alpha \tilde{V}_{12}^- \phi^- \\ & - \tilde{V}_{21}^- \phi + (z + \beta) \tilde{V}_{21}^- - \tilde{V}_{22}^- (\hbar z - 1) \alpha. \end{aligned} \quad (4.36)$$

Hence,

$$\begin{aligned} & (\phi^- + (\phi - (z + \beta)) S^-) \left( -\phi_{t_r} + \tilde{V}_{11} \phi + \tilde{V}_{12} \phi^2 - \tilde{V}_{21} - \tilde{V}_{22} \phi \right) \\ & = -(z + \beta) \tilde{V}_{22}^- \phi^- + \tilde{V}_{11} (z + \beta) \phi^- + (z + \beta)^2 \tilde{V}_{12} \phi^- \\ & = 0, \end{aligned} \quad (4.37)$$

using (4.9). Solving the first-order difference equation (4.37) then yields

$$\begin{aligned}
& -\phi_{t_r} + \tilde{V}_{11}\phi + \tilde{V}_{12}\phi^2 - \tilde{V}_{21} - \tilde{V}_{22}\phi \\
& = E(P, t_r) \times \begin{cases} \prod_{n'=1}^n B(P, n', t_r), & n \in \mathbb{N}, \\ 1, & n = 0, \\ \prod_{n'=0}^{n+1} B(P, n', t_r)^{-1}, & -n \in \mathbb{N}, \end{cases} \quad (4.38)
\end{aligned}$$

where

$$B(P, n', t_r) = \frac{\phi(P, n', t_r) - (z + \beta(n', t_r))}{\phi^-(P, n', t_r)}, \quad (n', t_r) \in \mathbb{Z} \times \mathbb{R}$$

and  $E(\cdot, t_r)$  is some  $n$ -independent meromorphic function on  $\mathcal{K}_p$ . The asymptotic behavior of  $\phi(P, n, t_r)$  in (3.33) then yields (for  $t_r \in \mathbb{R}$  fixed)

$$B(P) \underset{\zeta \rightarrow 0}{=} \frac{-1}{\hbar\alpha} \zeta^{-1} + O(1) \quad \text{as } P \rightarrow P_{\infty+}.$$

Comparing the order of both sides in (4.38) and taking  $n > 0$  sufficiently large, one finds contradiction unless  $E = 0$ . This proves (4.24). To prove (4.31) we rewrite (4.24) as

$$\begin{aligned}
\phi_{t_r} &= \left( -\tilde{V}_{11} + \tilde{V}_{11}^+ + (z + \beta^+) \tilde{V}_{12}^+ - \tilde{V}_{12}\phi + \frac{(\hbar z - 1)\alpha^+ \tilde{V}_{12}^+}{\phi} \right) \phi \\
&= \left( -\tilde{V}_{11} + \tilde{V}_{11}^+ + (z + \beta^+) \tilde{V}_{12}^+ - \tilde{V}_{12}\phi + \tilde{V}_{12}^+ \phi^+ - (z + \beta^+) \tilde{V}_{12}^+ \right) \phi \quad (4.39) \\
&= \left( \tilde{V}_{11}^+ + \tilde{V}_{12}^+ \phi^+ - \tilde{V}_{11} - \tilde{B}_{12}\phi \right) \phi
\end{aligned}$$

Abbreviating

$$\Delta(n_0, t_r) = \int_{t_{0,r}}^{t_r} \left( \tilde{V}_{11}(z, n_0, s) + \tilde{V}_{12}(z, n_0, s)\phi(P, n_0, s) \right) ds$$

one computes for  $n \geq n_0 + 1$ ,

$$\begin{aligned}
\psi_{1,t_r} &= \left( \exp(\Delta) \prod_{n'=n_0}^{n-1} \phi(n') \right)_{t_r} \\
&= \Delta_{t_r} \psi_1 + \exp(\Delta) \sum_{n'=n_0}^{n-1} \phi_{t_r}(n') \prod_{n'' \neq n'} \phi(n'')
\end{aligned}$$

$$\begin{aligned}
&= \left( \tilde{V}_{11}(z, n_0, t_r) + \tilde{V}_{12}(z, n_0, t_r) \phi(P, n_0, t_r) \right) \psi_1 \\
&+ \exp(\Delta) \sum_{n'=n_0}^{n-1} \left( \tilde{V}_{12}^+(n') \phi^+(n') + \tilde{V}_{11}^+(n') \right. \\
&\quad \left. - \tilde{V}_{12}(n') \phi(n') - \tilde{V}_{11}(n') \right) \phi(n') \prod_{n'' \neq n'} \phi(n'') \\
&= (\tilde{V}_{11} + \tilde{V}_{12} \phi) \psi_1.
\end{aligned} \tag{4.40}$$

The case  $n \leq n_0$  is handled analogously. By (4.28) and (4.40),

$$\begin{aligned}
\psi_{2,t_r} &= \phi_{t_r} \psi_1 + \phi \psi_{1,t_r} \\
&= [(-\tilde{V}_{11} + \tilde{V}_{12}) \phi - \tilde{V}_{12} \phi^2 + (\hbar z - 1) \alpha^+ \tilde{V}_{12}^+] \psi_1 + \phi [\tilde{V}_{11} \psi_1 + \tilde{V}_{12} \psi_2] \\
&= \tilde{V}_{21} \psi_1 + \tilde{V}_{22} \psi_2
\end{aligned} \tag{4.41}$$

Combining (4.40) with (4.41) then yields (4.31).

That  $\psi_1(P, n, n_0, t_0, t_r)$  is meromorphic on  $\mathcal{K}_p \setminus \{P_{\infty \pm}\}$  if  $V_{12}(\cdot, n_0, t_r)$  has only simple zeros distinct from  $P_{\hbar}$  is a consequence of (4.15)-(4.17), (4.19), (4.42) and of

$$\tilde{V}_{12} \phi \underset{P \rightarrow \hat{\mu}_j(n_0, s)}{=} \partial_s \ln(V_{12}(z, n_0, s)) + O(1), \quad \text{as } z \rightarrow \mu_j(n_0, s). \quad \square$$

Next we consider the  $t_r$ -dependence of  $V_{11}, V_{12}V_{21}$ .

**Lemma 4.3.** Assume Hypothesis 4.1 and suppose that (4.2), (4.3) hold. In addition, let  $(z, n, t_r) \in \mathbb{C} \times \mathbb{Z} \times \mathbb{R}$ . Then

$$V_{12,t_r} = (\tilde{V}_{11} - \tilde{V}_{22})V_{12} - 2V_{11}\tilde{V}_{12}, \tag{4.42}$$

$$V_{11,t_r} = \tilde{V}_{12}V_{21} - \tilde{V}_{21}V_{12}, \tag{4.43}$$

$$V_{21,t_r} = 2V_{11}\tilde{V}_{21} + (\tilde{V}_{22} - \tilde{V}_{11})V_{21}, \tag{4.44}$$

In particular, (4.42)-(4.44) are equivalent to

$$V_{p,t_r} = [\tilde{V}_r, V_p] \tag{4.45}$$

and the spectral curve  $\mathcal{K}_p$  defined by (2.31) and (4.12) is  $t_r$ -independent.

**Proof.** To prove (4.42) one first differentiates equation (4.27)

$$\phi_{t_r}(P) - \phi_{t_r}(P^*) = -\frac{yV_{12,t_r}}{V_{12}^2(z)}.$$

The time derivative of  $\phi$  given in (4.24) and (4.26) yields

$$\begin{aligned}\phi_{t_r}(P) - \phi_{t_r}(P^*) &= (-\tilde{V}_{11} + \tilde{V}_{22})(\phi(P) - \phi(P^*)) \\ &\quad - \tilde{V}_{12}(\phi(P) + \phi(P^*))(\phi(P) - \phi(P^*)) \\ &= (-\tilde{V}_{11} + \tilde{V}_{22})y/V_{12} + 2\tilde{V}_{12}V_{11}y/V_{12}^2,\end{aligned}$$

and hence

$$V_{12,t_r} = (\tilde{V}_{11} - \tilde{V}_{22})V_{12} - 2V_{11}\tilde{V}_{12}.$$

Similarly, starting from (4.26)

$$\phi_{t_r}(P) + \phi_{t_r}(P^*) = -2\frac{V_{11,t_r}}{V_{12}} + 2\frac{V_{11}V_{12,t_r}}{V_{12}^2}$$

yields (4.43). Moreover,

$$\begin{aligned}V_{21,t_r} &= (\hbar z - 1)\alpha_{t_r}^+ V_{12}^+ + (\hbar z - 1)\alpha^+ V_{12,t_r}^+ \\ &= -(\hbar z - 1)V_{12}^+ \left( \alpha^+ \tilde{V}_{11} + (z + \beta^+)\alpha^+ \tilde{V}_{12}^+ \right. \\ &\quad \left. - \alpha^+ \tilde{V}_{22}^+ \right) + (\hbar z - 1)\alpha^+ (\tilde{V}_{11}^+ V_{12}^+ - \tilde{V}_{22}^+ V_{12}^+ \\ &\quad - 2V_{11}^+ \tilde{V}_{12}^+) \\ &= -V_{21}\tilde{V}_{11} - (z + \beta^+)V_{12}^+ \tilde{V}_{21} + \tilde{V}_{11}^+ - 2V_{11}^+ \tilde{V}_{21} \\ &= 2V_{11}\tilde{V}_{21} + (\tilde{V}_{22} - \tilde{V}_{11})V_{21},\end{aligned}$$

using (4.7), (4.9) and (4.10). Finally, by (4.42)–(4.44), differentiating (4.12) with respect to  $t_r$  then yields

$$R_{2p+2,t_r} = -2V_{11}V_{11,t_r} - V_{12,t_r}V_{21} - V_{12}V_{21,t_r} = 0. \quad (4.46)$$

□

Next we turn to the Dubrovin equation for the time variation of the zeros  $\mu_j$  of  $V_{12}$  governed by the  $\widetilde{\text{RT}}_r$  flow.

**Lemma 4.4.** Assume Hypothesis 4.1 and suppose that (4.2), (4.3) hold on  $\mathbb{Z} \times \mathcal{I}_\mu$  with  $\mathcal{I}_\mu \subseteq \mathbb{R}$  an open interval. In addition, assume that the zeros  $\mu_j$ ,  $j = 1, \dots, p$ , of  $V_{12}$  remain distinct on  $\mathbb{Z} \times \mathcal{I}_\mu$ . Then  $\{\hat{\mu}_j\}_{j=1,\dots,p}$  defined in (4.13) and (4.14) satisfy the following first-order system of differential equation on  $\mathbb{Z} \times \mathcal{I}_\mu$ ,

$$\mu_{j,t_r} = -\tilde{V}_{12}(\mu_j)y(\hat{\mu}_j) \prod_{\substack{j=1 \\ k \neq j}}^p (\mu_j - \mu_k)^{-1}, \quad j = 1, \dots, p, \quad (4.47)$$



with

$$\hat{\mu}_j(n, \cdot) \in C^\infty(\mathcal{I}_\mu, \mathcal{K}_p), \quad j = 1, \dots, p, \quad n \in \mathbb{Z}.$$

**Proof.** It suffices to consider (4.47) for  $\mu_{j,t_r}$ . Using the product representations for  $V_{12}$  in (4.6) and employing (4.13) and (4.42), one computes

$$\begin{aligned} V_{12,t_r}(\mu_j) &= -\mu_{j,t_r} \prod_{\substack{j=1 \\ k \neq j}}^p (\mu_j - \mu_k) \\ &= -2V_{11}(\mu_j)\tilde{V}_{12}(\mu_j) = y(\hat{\mu}_j)\tilde{V}_{12}(\mu_j), \quad j = 1, \dots, p, \end{aligned}$$

proving (4.47).  $\square$

Since the stationary trace formulas for  $f_\ell$  in terms of symmetric functions of the zeros  $\mu_j$  of  $V_{12}$  in Lemma 3.2 extend line by line to the corresponding time-dependent setting, we next record their  $t_r$ -dependent analogs without proof. For simplicity we again confine ourselves to the simplest cases only.

**Lemma 4.5.** Assume Hypothesis 4.1 and suppose that (4.2), (4.3) hold. Then

$$-\hbar(\alpha + \alpha^+) - \beta + \delta_1 = -\sum_{j=1}^p \mu_j. \quad (4.48)$$

Next, we turn to the asymptotic expansions of  $\phi$  and  $\psi_1$  in a neighborhood of  $P_{\infty\pm}$  and  $P_h$ . 0.4cm

**Lemma 4.6.** Assume Hypothesis 4.1 and suppose that (4.2), (4.3) hold. Moreover, let  $P = (z, y) \in \mathcal{K}_p \setminus \{P_{\infty\pm}, P_h\}$ ,  $(n, n_0, t_r, t_{0,r}) \in \mathbb{Z}^2 \times \mathbb{R}^2$ . Then  $\phi$  has the asymptotic behavior

$$\phi(P) \underset{\zeta \rightarrow 0}{=} \begin{cases} \zeta^{-1} + (\beta - \hbar\alpha) + O(\zeta), & P \rightarrow P_{\infty-}, \\ \hbar\alpha^+ + (\hbar^2\alpha^+\alpha + \alpha - \hbar\alpha\beta)\zeta + O(\zeta^2), & P \rightarrow P_{\infty+}, \end{cases} \quad \zeta = 1/z, \quad (4.49)$$

$$\phi(P) \underset{\zeta \rightarrow 0}{=} \frac{\hbar\alpha^+}{\hbar + \beta^+}\zeta + O(\zeta^2), \quad P \rightarrow P_h, \quad \zeta = z - 1/\hbar. \quad (4.50)$$

The component  $\psi_1$  of the Baker-Akhiezer vector  $\Psi$  has asymptotic behavior

$$\begin{aligned} \psi_1(P, n, n_0, t_r, t_{0,r}) \underset{\zeta \rightarrow 0}{=} & \exp \left( \pm \frac{1}{2} (t_r - t_{0,r}) \sum_{s=0}^{r+1} \tilde{\delta}_{r+1-s} \zeta^{-(s+1)} (1 + O(\zeta)) \right) \\ & \times \begin{cases} \zeta^{n_0-n} (1 + O(\zeta)), & \text{as } P \rightarrow P_{\infty-}, \\ \Gamma(\hbar\alpha^+) (1 + O(\zeta)) \\ \times \exp \left( \int_{t_{0,r}}^{t_r} \left( \sum_{j=0}^{r+1} \tilde{\delta}_{r+1-j} \hat{f}_{j+2}(n_0, s) \right) ds \right), & \text{as } P \rightarrow P_{\infty+}, \end{cases} \end{aligned} \quad (4.51)$$

$$\begin{aligned} \psi_1(P, n, n_0, t_r, t_{0,r}) \underset{\zeta \rightarrow 0}{=} & \Gamma\left(\frac{\hbar\alpha^+}{\hbar + \beta^+}\right) \zeta^{n-n_0} (1 + O(\zeta)) \exp \left( \int_{t_0}^{t_r} \tilde{V}_{11}(\hbar, n_0, s) ds \right), \\ & P \rightarrow P_h. \end{aligned} \quad (4.52)$$

**Proof.** Since by the definition of  $\phi$  in (4.15) and (4.16) the time parameter  $t_r$  can be viewed as an additional but fixed parameter, the asymptotic behavior of  $\phi$  remains the same as in Lemma 3.3. Similarly, also the asymptotic behavior of  $\psi_1(P, n, n_0, t_r, t_r)$  is derived in an identical fashion to that in Lemma 3.3. This proves (4.51) for  $t_{0,r} = t_r$ , that is,

$$\psi_1(P, n, n_0, t_r, t_r) \underset{\zeta \rightarrow 0}{=} \begin{cases} \zeta^{n_0-n} (1 + O(\zeta)), & P \rightarrow P_{\infty-}, \\ \Gamma(\hbar\alpha^+) (1 + O(\zeta)), & P \rightarrow P_{\infty+}, \end{cases} \quad \zeta = 1/z, \quad (4.53)$$

It remain to investigate

$$\psi_1(P, n, n_0, t_r, t_{0,r}) = \exp \left( \int_{t_{0,r}}^{t_r} \left( \tilde{V}_{11}(z, n_0, s) + \tilde{V}_{12}(z, n_0, s) \right) \phi(P, n_0, s) ds \right). \quad (4.54)$$

Next, it is convenient to introduce the homogenous representations of  $V_{11}, V_{12}$  defined by vanishing all of their integration constants  $\delta_\ell$ , that is,

$$\begin{aligned} \hat{V}_{11} &= \sum_{j=0}^{p+1} \hat{g}_{p+1-\ell} z^\ell + \hat{f}_{r+2} = V_{11}|_{\delta_0=1, \delta_j=0, j=1, \dots, p+1}, \quad p \in \mathbb{N}, \\ \hat{V}_{12} &= \sum_{j=0}^{p+1} \hat{f}_{p+1-\ell} z^\ell = V_{12}|_{\delta_0=1, \delta_j=0, j=1, \dots, p+1}, \quad p \in \mathbb{N}. \end{aligned} \quad (4.55)$$

In order to avoid confusion about notation, we relabel  $V_{11}, V_{12}$  by  $V_{11}^{(p)}, V_{12}^{(p)}$  to represent the polynomials associated with the  $p$ th stationary Ruijsenaars-Toda equation (2.24). Then we have

$$\begin{aligned} V_{11} &= V_{11}^{(p+1)} = \sum_{j=0}^{p+1} \delta_{p+1-j} \widehat{V}_{11}^{(j)}, \quad V_{12} = V_{12}^{(p+1)} = \sum_{j=0}^{p+1} \delta_{p+1-j} \widehat{V}_{12}^{(j)}, \\ \widetilde{V}_{11} &= \widetilde{V}_{11}^{(r+1)} = \sum_{j=0}^{r+1} \widetilde{\delta}_{r+1-j} \widehat{V}_{11}^{(j)}, \quad \widetilde{V}_{12} = \widetilde{V}_{12}^{(r+1)} = \sum_{j=0}^{r+1} \widetilde{\delta}_{r+1-j} \widehat{V}_{12}^{(j)}. \end{aligned} \quad (4.56)$$

Focusing on the homogeneous first, one computes as  $P \rightarrow P_{\infty\pm}$ ,

$$\begin{aligned} \widehat{V}_{11}^{(j)} + \widehat{V}_{12}^{(j)} \phi &= \widehat{V}_{11}^{(j)} + \widehat{V}_{12}^{(j)} \frac{y/2 - V_{11}^{(p)}}{V_{12}^{(p)}} \\ &= \widehat{V}_{11}^{(j)} + \widehat{V}_{12}^{(j)} \frac{1/2 - V_{11}^{(p)}/y}{V_{12}^{(p)}/y} \\ &= \sum_{\ell=0}^{j+1} \hat{g}_{j+1-\ell} z^\ell + \hat{f}_{j+2} + \left( \sum_{\ell=0}^{j+1} \hat{f}_{j+1-\ell} z^\ell \right) \frac{1/2 - V_{11}^{(p)}/y}{V_{12}^{(p)}/y} \\ &= \pm \frac{1}{2} \zeta^{-(j+1)} + \begin{cases} \hat{f}_{j+2} + O(\zeta), & P \rightarrow P_{\infty+}, \\ 0 + O(\zeta), & P \rightarrow P_{\infty-}, \end{cases} \quad \zeta = 1/z. \end{aligned}$$

One infers from (4.56)

$$\widetilde{V}_{11} + \widetilde{V}_{12} \phi = \pm \frac{1}{2} \sum_{j=0}^{r+1} \widetilde{\delta}_{r+1-j} \zeta^{-j} + \begin{cases} \sum_{j=0}^{r+1} \widetilde{\delta}_{r+1-j} \hat{f}_{j+2} + O(\zeta), & P \rightarrow P_{\infty+}, \\ O(\zeta), & P \rightarrow P_{\infty-}, \end{cases} \quad \zeta = 1/z. \quad (4.57)$$

Insertion of (4.57) into (4.54) then proves (4.51). Finally, (4.52) follows from (4.19) and (4.50).  $\square$

Next, we turn to the principal result of this section, the representation of  $\phi, \psi_1, \alpha$  and  $\beta$  in terms of the Riemann theta function associated with  $\mathcal{K}_p$ , assuming  $p \in \mathbb{N}$  for the remainder of this section.

Let  $\omega_{P_{\infty\pm}, q}^{(2)}$  be the normalized differentials of the second kind with a unique pole at  $P_{\infty\pm}$ , respectively, and principal parts

$$\omega_{P_{\infty\pm}, q}^{(2)} \underset{\zeta \rightarrow 0}{=} (\zeta^{-2-q} + O(1)) d\zeta, \quad P \rightarrow P_{\infty\pm}, \quad \zeta = 1/z, \quad q \in \mathbb{N}_0, \quad (4.58)$$

with vanishing  $a$ -periods,

$$\int_{a_j} \omega_{P_{\infty\pm},q}^{(2)} = 0, \quad j = 1, \dots, p.$$

Moreover, we define

$$\tilde{\Omega}_r^{(2)} = \frac{1}{2} \left( \sum_{j=1}^{r+1} j \tilde{\delta}_{r+1-j} \left( \omega_{P_{\infty+},j-1}^{(2)} - \omega_{P_{\infty-},j-1}^{(2)} \right) \right) \quad (4.59)$$

and corresponding vector of  $b$ -periods of  $\tilde{\Omega}_r^{(2)}/(2\pi i)$  is then denoted by

$$\underline{\tilde{U}}_r^{(2)} = \left( \tilde{U}_{r,1}^{(2)}, \tilde{U}_{r,2}^{(2)}, \dots, \tilde{U}_{r,p}^{(2)} \right), \quad \tilde{U}_{r,j}^{(2)} = \frac{1}{2\pi i} \int_{bj} \tilde{\Omega}_r^{(2)}, \quad j = 1, 2, \dots, p.$$

Finally, we abbreviate

$$\tilde{\Omega}_r^{\infty-} = \lim_{P \rightarrow P_{\infty-}} \left( \int_{Q_0}^P \tilde{\Omega}_r^{(2)} - \frac{1}{2} \sum_{s=0}^{r+1} \tilde{\delta}_{r+1-s} \zeta^{-(s+1)} \right)$$

and

$$\omega_0 = \lim_{P \rightarrow P_{\infty-}} \left( \int_{Q_0}^P \omega_{P_h P_{\infty-}}^{(3)} + \ln \zeta \right).$$

**Theorem 4.7.** Assume Hypothesis 4.1 and suppose that (4.2), (4.3) hold. Moreover, let  $P = (z, y) \in \mathcal{K}_p \setminus \{P_{\infty\pm}, P_h\}$ ,  $(n, n_0, t_r, t_{0,r}) \in \mathbb{Z}^2 \times \mathbb{R}^2$ , Then for each  $(n, t_r) \in \mathbb{Z} \times \mathbb{R}$ ,  $\mathcal{D}_{\underline{\hat{\mu}}(n, t_r)}$  is nonspecial. Moreover,

$$\phi(P, n) = C(n, t_r) \frac{\theta(\underline{z}(P, \underline{\hat{\mu}}^+(n, t_r)))}{\theta(\underline{z}(P, \underline{\hat{\mu}}(n, t_r)))} \exp \left( \int_{Q_0}^P \omega_{P_h P_{\infty-}}^{(3)} \right), \quad (4.60)$$

$$\begin{aligned} \psi_1(P, n, n_0, t_r, t_{0,r}) = & C(n, n_0, t_r, t_{0,r}) \frac{\theta(\underline{z}(P, \underline{\hat{\mu}}(n, t_r)))}{\theta(\underline{z}(P, \underline{\hat{\mu}}(n_0, t_{0,r})))} \exp \left( (n - n_0) \int_{Q_0}^P \omega_{P_h P_{\infty-}}^{(3)} \right. \\ & \left. - (t_r - t_{0,r}) \int_{Q_0}^P \tilde{\Omega}_r^{(2)} \right), \end{aligned} \quad (4.61)$$

where

$$C(n, t_r) = c_0^{-1} \frac{\theta(\underline{z}(P_{\infty-}, \underline{\hat{\mu}}(n, t_r)))}{\theta(\underline{z}(P_{\infty-}, \underline{\hat{\mu}}^+(n, t_r)))} \quad (4.62)$$

and

$$C(n, n_0, t_r, t_{0,r}) = \exp \left( (t_r - t_{0,r}) \tilde{\Omega}_r^{\infty-} - (n - n_0) \omega_0 \right) \times \frac{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}(n_0, t_{0,r})))}{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}(n, t_r)))}. \quad (4.63)$$

The Abel map linearizes the auxiliary divisor  $\mathcal{D}_{\hat{\underline{\mu}}(n, t_r)}$  in the sense that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\mu}}(n, t_r)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\mu}}(n_0, t_{0,r})}) - \underline{A}_{P_{\infty-}}(P_h)(n - n_0) - \tilde{U}_r^{(2)}(t_r - t_{0,r}) \quad (4.64)$$

and  $\alpha, \beta$  are the form of

$$\alpha^+(n, t_r) = \frac{c_1}{\hbar c_0} \frac{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}(n, t_r)))}{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}^+(n, t_r)))} \frac{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}^+(n, t_r)))}{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(n, t_r)))} \quad (4.65)$$

and

$$\begin{aligned} \beta(n, t_r) &= (c_1/c_0) \frac{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}^-(n, t_r)))}{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}(n, t_r)))} \frac{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(n, t_r)))}{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}^-(n, t_r)))} \\ &\quad - \frac{1}{4} \sum_{m=0}^{2p+1} E_m - \frac{1}{2} \hbar^{-1} + \frac{1}{2} \sum_{j=1}^p \lambda_j \\ &\quad + \frac{\partial}{\partial \omega_j} \ln \left( \frac{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}^+(n, t_r)) + \underline{\omega})}{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}(n, t_r)) + \underline{\omega})} \right) \Big|_{\underline{\omega}=0}. \end{aligned} \quad (4.66)$$

Here  $c_0, c_1 \in \mathbb{C}$  are integration constants.

**Proof.** As in Theorem 3.5 one concludes that  $\phi(P, n, t_r)$  has the form (4.60) and that for  $t_{0,r} = t_r$ ,  $\psi_1(P, n, n_0, t_r, t_{0,r})$  is of the form

$$\begin{aligned} \psi_1(P, n, n_0, t_r, t_r) &= C(n, n_0, t_r, t_r) \frac{\theta(\underline{z}(P, \hat{\underline{\mu}}(n, t_r)))}{\theta(\underline{z}(P, \hat{\underline{\mu}}(n_0, t_r)))} \\ &\quad \times \exp \left( (n - n_0) \int_{Q_0}^P \omega_{P_0 P_{\infty+}}^{(3)} \right). \end{aligned}$$

To discuss  $\psi_1(P, n, n_0, t_r, t_{0,r})$  we recall (4.22), that is,

$$\psi_1(P, n, n_0, t_r, t_{0,r}) = \psi_1(P, n_0, n_0, t_r, t_{0,r}) \psi_1(P, n, n_0, t_r, t_r), \quad (4.67)$$

and hence remaining to be studied is

$$\psi_1(P, n_0, n_0, t_r, t_{0,r}) = \exp \left( \int_{t_{0,r}}^{t_r} \tilde{V}_{11}(z, n_0, s) + \tilde{V}_{12}(z, n_0, s) \phi(P, n_0, s) \right). \quad (4.68)$$

Introducing  $\hat{\psi}_1(P)$  on  $\mathcal{K}_p \setminus \{P_{\infty\pm}\}$  by

$$\begin{aligned} \hat{\psi}_1 &= C(n_0, n_0, t_r, t_{0,r}) \frac{\theta(\underline{z}(P, \hat{\mu}(n_0, t_r)))}{\theta(\underline{z}(P, \hat{\mu}(n_0, t_{0,r})))} \\ &\quad \times \exp\left(-(t_r - t_{0,r}) \int_{Q_0}^P \tilde{\Omega}_r^{(2)}\right), \end{aligned} \quad (4.69)$$

we intend to prove that

$$\begin{aligned} \psi_1(P, n_0, n_0, t_r, t_{0,r}) &= \hat{\psi}_1(P, n_0, t_r, t_{0,r}) \\ P &\in \mathcal{K}_p \setminus \{P_{\infty\pm}\}, \quad n_0 \in \mathbb{Z}, \quad t_r, t_{0,r} \in \mathbb{R}, \end{aligned} \quad (4.70)$$

for an appropriate choice of the normalization constant  $C(n_0, n_0, t_r, t_{0,r})$  in (4.69). We start by noting that a comparison of (4.51), (4.52), (4.59), (4.61) shows that  $\psi_1$  and  $\hat{\psi}_1$  have the same essential singularities at  $P_{\infty\pm}$ . Thus, we turn to the local behavior of  $\psi_1$  and  $\hat{\psi}_1$ . By (4.69)  $\hat{\psi}_1$  has zeros and poles at  $\hat{\mu}(n_0, t_r)$  and  $\hat{\mu}(n_0, t_{0,r})$ . Similarly, by (4.68),  $\hat{\psi}_1$  has zeros and poles only at poles of  $\phi(P, n_0, s)$ ,  $s \in [t_{0,r}, t_r]$  (resp.,  $s \in [t_r, t_{0,r}]$ ). In the following we temporarily restrict  $t_{0,r}$  and  $t_r$  to a sufficiently small nonempty interval  $I \subseteq \mathbb{R}$  and pick  $n_0 \in \mathbb{Z}$  such that for all  $s \in I$ ,  $\mu_j(n_0, s) \neq \mu_k(n_0, s)$  for all  $j \neq k, j, k = 1, \dots, p$ . One computes

$$\begin{aligned} &\psi_1(P, n_0, n_0, t_r, t_{0,r}) \\ &= \exp\left(\int_{t_{0,r}}^{t_r} ds \left(\tilde{V}_{11}(z, n_0, s) + \tilde{V}_{12}(z, n_0, s) \frac{y/2 - V_{11}(z, n_0, s)}{V_{12}(z, n_0, s)}\right)\right) \\ &\stackrel{=}{=} \exp\left(\int_{t_{0,r}}^{t_r} ds \left(\frac{\tilde{V}_{12}(\mu_j(n_0, s)) y(\hat{\mu}_j(n_0, s))}{(z - \mu_j(n_0, s)) \prod_{\substack{k=1 \\ k \neq j}}^p (\mu_j(n_0, s) - \mu_k(n_0, s))} + O(1)\right)\right) \\ &\stackrel{=}{=} \exp\left(\int_{t_0}^{t_r} ds \left(\frac{-\mu_{j,s}(n_0, s)}{z - \mu_j(n_0, s)} + O(1)\right)\right) \\ &\stackrel{=}{=} \exp\left(\int_{t_0}^{t_r} ds \left(\frac{\partial}{\partial s} \ln(\mu_j(n_0, s) - z) + O(1)\right)\right). \end{aligned} \quad (4.71)$$

Restricting  $P$  to a sufficiently small neighborhood  $\mathcal{U}_j(n_0)$  of  $\{\hat{\mu}_j(n_0, s) \in \mathcal{K}_p | s \in [t_{0,r}, t_r] \subseteq I\}$  such that  $\hat{\mu}_k(n_0, s) \notin \mathcal{U}_j(n_0)$  for all  $s \in [t_{0,r}, t_r] \subseteq I$  and

all  $k \in \{1, \dots, p\} \setminus \{j\}$ , (4.69) and (4.71) imply

$$\begin{aligned} & \psi_1(P, n_0, n_0, t_r, t_{0,r}) \\ &= \begin{cases} (\mu_j(n_0, t_r) - z)O(1), & \text{as } P \rightarrow \hat{\mu}_j(n_0, t_r) \neq \hat{\mu}_j(n_0, t_{0,r}), \\ O(1), & \text{as } P \rightarrow \hat{\mu}_j(n_0, t_r) = \hat{\mu}_j(n_0, t_{0,r}), \\ (\mu_j(n_0, t_{0,r}) - z)^{-1}O(1), & \text{as } P \rightarrow \hat{\mu}_j(n_0, t_{0,r}) \neq \hat{\mu}_j(n_0, t_r), \end{cases} \\ & \quad P = (z, y) \in \mathcal{K}_p, \quad (4.72) \end{aligned}$$

with  $O(1) \neq 0$ . Thus  $\psi_1$  and  $\hat{\psi}_1$  have the same local behavior and identical essential singularities at  $P_{\infty\pm}$ . Hence  $\psi_1$  and  $\hat{\psi}_1$  coincide up to a multiple constant (which may depend on  $n_0, t_r, t_{0,r}$ ). By continuity with respect to divisors this extend to all  $n_0 \in \mathbb{Z}$  since by hypothesis  $\mathcal{D}_{\hat{\mu}(n,s)}$  remain nonspecial for all  $(n, s) \in \mathbb{Z} \times \mathbb{R}$ . Moreover, since by (4.68), for fixed  $P$  and  $n_0$ ,  $\psi_1(P, n_0, n_0, \cdot, t_{0,r})$  is entire in  $t_r$  (and this argument is symmetric in  $t_r$  and  $t_{0,r}$ ), (4.70) holds for all  $t_r, t_{0,r} \in \mathbb{R}$  (for an appropriate choice of  $C(n_0, n_0, t_r, t_{0,r})$ ). Together with (4.67), this proves (4.61) for all  $(n, t_r), (n_0, t_{0,r}) \in \mathbb{Z} \times \mathbb{R}$ .

To determine the constant  $C(n, n_0, t_r, t_{0,r})$  one compares the asymptotic expansions of  $\psi_1(P, n, n_0, t_r, t_{0,r})$  for  $P \rightarrow P_{\infty-}$  in (4.51) and (4.61)

$$\begin{aligned} C(n, n_0, t_r, t_{0,r}) &= \exp \left( (t_r - t_{0,r}) \tilde{\Omega}_r^{\infty-} - (n - n_0) \omega_0 \right) \\ &\quad \times \frac{\theta(\underline{z}(P_{\infty-}, \hat{\mu}(n_0, t_{0,r})))}{\theta(\underline{z}(P_{\infty-}, \hat{\mu}(n, t_r)))}. \end{aligned}$$

Finally, (4.64) follows from

$$\begin{aligned} & \frac{\partial}{\partial t_r} \alpha_{Q_0, \ell}(\mathcal{D}_{\hat{\mu}(n, t_r)}) \\ &= \frac{\partial}{\partial t_r} \sum_{j=1}^p \int_{Q_0}^{\hat{\mu}_j(n, t_r)} \omega_\ell \\ &= \sum_{j=1}^p \omega_\ell(\hat{\mu}_j) \mu_{j, t_r} \\ &= \sum_{j=1}^p \left( \sum_{k=1}^p c_\ell(k) \frac{\mu_j^{k-1}}{y(\hat{\mu}_j(n, t_r))} \right) \left( -\tilde{V}_{12}(\mu_j(n, t_r)) y(\hat{\mu}_j(n, t_r)) \prod_{\substack{k=1 \\ k \neq j}}^p (\mu_j - \mu_k)^{-1} \right) \\ &= \sum_{j=1}^p \left( \sum_{k=1}^p c_\ell(k) \frac{\mu_j^{k-1}}{\prod_{k=1, k \neq j}^p (\mu_j - \mu_k)} \right) \left( -\tilde{V}_{12}(\mu_j(n, t_r)) \right) \end{aligned}$$

$$\begin{aligned}
&= - \sum_{k=1}^p c_\ell(k) \sum_{j=1}^p \frac{\mu_j^{k-1}}{\prod_{k=1, k \neq j}^p (\mu_j - \mu_k)} \left( \sum_{s=0}^{r+1} \tilde{\delta}_{r+1-s} \left( \sum_{t=\max\{0, s-p\}}^s \hat{c}_t(\underline{E}) \Psi_{s-t}^{(j)}(\underline{\mu}) \right) \right) \\
&= - \sum_{k=1}^p \sum_{s=0}^r c_\ell(k) \tilde{\delta}_{r-s} \hat{c}_{k+s-p}(\underline{E}) \\
&= - \tilde{U}_r^{(2)},
\end{aligned}$$

where we use the interpolation representation of  $\tilde{V}_{12}$  in appendix A (cf. (4.79)) and

$$\begin{aligned}
\omega_j &= \pm \sum_{j=1}^p c_j(k) \frac{\zeta^{p-j}}{\left( \prod_{m=0}^{2p+1} (1 - E_m \zeta) \right)^{\frac{1}{2}}} d\zeta \\
&= \pm \left( \sum_{q=0}^{\infty} \sum_{k=1}^p c_j(k) \hat{c}_{k-p+q}(\underline{E}) \zeta^q \right) d\zeta, \\
\tilde{U}_{r,j}^{(2)} &= \frac{1}{2\pi i} \int_{b_j} \tilde{\Omega}_r^{(2)} \\
&= \frac{1}{2\pi i} \left[ \frac{1}{2} \sum_{s=1}^{r+1} s \tilde{\delta}_{r+1-s} \left( \int_{b_j} \omega_{P_{\infty+}, s-1}^{(2)} - \int_{b_j} \omega_{P_{\infty-}, s-1}^{(2)} \right) \right] \\
&= \sum_{s=1}^{r+1} \tilde{\delta}_{r+1-s} \sum_{k=1}^p c_\ell(k) \hat{c}_{k-p+s}(\underline{E}). \quad \square
\end{aligned}$$

## Appendix A: The Lagrange Interpolation Representation of $\tilde{V}_{12}(\mu_j(n, t_r))$

Introducing the notation in [37, 38],

$$\begin{aligned}
\Psi_k(\underline{\mu}) &= (-1)^k \sum_{\underline{\ell} \in \mathcal{S}_k} \mu_{\ell_1} \dots, \mu_{\ell_k}, \\
\mathcal{S}_k &= \{ \underline{\ell} = (\ell_1, \dots, \ell_k) \in \mathbb{N}^k \mid \ell_1 < \dots < \ell_k \leq p \}, \quad k = 1, \dots, p, \\
\Phi_k^{(j)}(\underline{\mu}) &= (-1)^k \sum_{\underline{\ell} \in \tau_k^{(j)}} \mu_{\ell_1} \dots, \mu_{\ell_k}, \\
\tau_k^{(j)} &= \{ \underline{\ell} = (\ell_1, \dots, \ell_k) \in \mathbb{N}^k \mid \ell_1 < \dots < \ell_k \leq p \quad \ell_m \neq j \}, \\
&\quad k = 1, \dots, p-1, \quad j = 1, \dots, p.
\end{aligned}$$



and the formula

$$\sum_{\ell=0}^k \Psi_{k-\ell}(\underline{\mu}) \mu_j^\ell = \Phi_k^{(j)}(\underline{\mu}), \quad k = 0, \dots, n, \quad j = 1, \dots, n, \quad (4.73)$$

one finds

$$V_{12}(z) = \sum_{s=0}^{p+1} f_{p+1-s} z^s = \prod_{j=1}^p (z - \mu_j) = \sum_{\ell=0}^p \Psi_{p-\ell}(\underline{\mu}) z^\ell$$

and

$$f_\ell = \Psi_{\ell-1}(\underline{\mu}), \quad \ell = 0, \dots, p+1. \quad (\text{define } \Psi_{-1}(\underline{\mu}) = 0)$$

In the case  $r < p$ ,

$$\begin{aligned} \widehat{V}_{12} &= \sum_{s=0}^{r+1} \widehat{f}_{r+1-s} z^s \\ &= \sum_{s=0}^{r+1} \left( \sum_{k=0}^{\min\{r+1-s, p+1\}} \widehat{c}_{r+1-s-k}(\underline{E}) f_k \right) z^s \\ &= \sum_{s=0}^{r+1} \left( \sum_{k=0}^{r+1-s} \widehat{c}_{r+1-s-k}(\underline{E}) f_k \right) z^s \\ &= \sum_{s=0}^{r+1} \left( \sum_{k=0}^{r+1-s} \widehat{c}_{r+1-s-k}(\underline{E}) \Psi_{k-1}(\underline{\mu}) \right) z^s \\ &= \sum_{s=0}^{r+1} \widehat{c}_s(\underline{E}) \sum_{t=0}^{r+1-s} \Psi_{r-s-t}(\underline{\mu}) z^t \\ &= \sum_{s=0}^r \widehat{c}_s(\underline{E}) \sum_{t=0}^{r-s} \Psi_{r-s-t}(\underline{\mu}) z^t. \end{aligned} \quad (4.74)$$

Using (4.73), we have

$$\begin{aligned} \widehat{V}_{12}(\mu_j) &= \sum_{s=0}^r \widehat{c}_s(\underline{E}) \sum_{t=0}^{r-s} \Psi_{r-s-t}(\underline{\mu}) \mu_j^t \\ &= \sum_{s=0}^r \widehat{c}_s(\underline{E}) \Psi_{r-s}^{(j)}(\underline{\mu}). \end{aligned} \quad (4.75)$$

In the case  $r > p$ ,

$$\begin{aligned}
& \widehat{V}_{12}(z) \\
&= \sum_{s=0}^{r+1} \widehat{f}_{r+1-s} z^s \\
&= \sum_{s=0}^{r+1} \left( \sum_{k=0}^{\min\{r+1-s, p+1\}} \widehat{c}_{r+1-s-k}(\underline{E}) f_k \right) z^s \\
&= \sum_{s=0}^{r-p} \sum_{k=0}^{p+1} \widehat{c}_{r+1-s-k}(\underline{E}) \Psi_{k-1}(\underline{\mu}) z^s + \sum_{s=r-p+1}^{r+1} \sum_{k=0}^{r+1-s} \widehat{c}_{r+1-s-k}(\underline{E}) \Psi_{k-1}(\underline{\mu}) z^s \\
&= \sum_{s=0}^{r-p} \sum_{k=0}^{p+1} \widehat{c}_{r+1-s-k}(\underline{E}) \Psi_{k-1}(\underline{\mu}) z^s + \sum_{s=r-p+1}^{r+1} \sum_{k=0}^{p+1} \widehat{c}_{r+1-s-k}(\underline{E}) \Psi_{k-1}(\underline{\mu}) z^s \\
&= \sum_{k=0}^{p+1} \sum_{s=0}^{r+1} \widehat{c}_{r+1-s-k}(\underline{E}) \Psi_{k-1}(\underline{\mu}) z^s \\
&= \sum_{s=0}^{r+1} \sum_{k=0}^{p+1} \widehat{c}_{r+1-s-k}(\underline{E}) \Psi_{k-1}(\underline{\mu}) z^s \\
&= \sum_{s=0}^{r+1} \sum_{k=0}^{p+1} \widehat{c}_s(\underline{E}) \Psi_{k-1} z^{r+1-s-k} \\
&= \sum_{s=0}^{r-p} \widehat{c}_s(\underline{E}) \left( \sum_{k=0}^{p+1} \Psi_{k-1}(\underline{\mu}) z^{p+1-k} \right) z^{r-p-s} \\
&\quad + \sum_{s=r-p+1}^{r+1} \widehat{c}_s(\underline{E}) \left( \sum_{k=0}^{p+1} \Psi_{k-1}(\underline{\mu}) z^{r+1-s-k} \right) \\
&= \sum_{s=0}^{r-p} \widehat{c}_s(\underline{E}) (V_{12}(z)) z^{r-p-s} + \sum_{s=r-p+1}^{r+1} \widehat{c}_s(\underline{E}) \left( \sum_{k=0}^{r+1-s} \Psi_{k-1}(\underline{\mu}) z^{r+1-s-k} \right).
\end{aligned} \tag{4.76}$$

Then one finds

$$\begin{aligned}
\widetilde{V}_{12}(\mu_j) &= \sum_{s=r-p+1}^{r+1} \widehat{c}_s(\underline{E}) \left( \sum_{k=0}^{r+1-s} \Psi_{k-1}(\underline{\mu}) \mu_j^{r+1-s-k} \right) \\
&= \sum_{s=r-p+1}^{r+1} \widehat{c}_s(\underline{E}) \Psi_{r-s}^{(j)}(\underline{\mu}).
\end{aligned} \tag{4.77}$$

Combining (4.75) with (4.77) yields

$$\widehat{V}_{12}(z) = \sum_{s=\max\{0, r-p+1\}}^{r+1} \hat{c}_s(\underline{E}) \Psi_{r-s}^{(j)}(\underline{\mu}) = \sum_{s=\max\{0, r-p\}}^r \hat{c}_s(\underline{E}) \Psi_{r-s}^{(j)}(\underline{\mu}) \quad (4.78)$$

and hence

$$\begin{aligned} \widetilde{V}_{12}(\mu_j) &= \sum_{s=0}^{r+1} \widetilde{\delta}_{r+1-s} \widehat{V}_{12}^{(s)}(\mu_j) \\ &= \sum_{s=0}^{r+1} \widetilde{\delta}_{r+1-s} \left( \sum_{t=\max\{0, s-p\}}^s \hat{c}_t(\underline{E}) \Psi_{s-t}^{(j)}(\underline{\mu}) \right). \end{aligned} \quad (4.79)$$

## Appendix B: Asymptotic Spectral Parameter Expansions

Next, we turn to asymptotic expansions of various quantities in the case of the Ruijsenaars-Toda Hierarchy. Consider a fundamental system of solutions  $\Psi_{\pm}(z, \cdot) = (\psi_{1,\pm}(z, \cdot), \psi_{2,\pm}(z, \cdot))^{\top}$  of  $U(z)\Psi_{\pm}(z) = \Psi_{\pm}(z)$  for  $z \in \mathbb{C}$  (or in some subdomain of  $\mathbb{C}$ ), with  $U$  given by (2.3), such that

$$\det(\Psi_{-}(z), \Psi_{+}(z)) \neq 0.$$

Introducing

$$\phi_{\pm} = \frac{\psi_{2,\pm}(z, n)}{\psi_{1,\pm}(z, n)}, \quad z \in \mathbb{C}, \quad n \in \mathbb{N}, \quad (4.80)$$

then  $\phi_{\pm}$  satisfy the Riccati-type equation

$$\phi_{\pm} \phi_{\pm}^{-} - (z + \beta) \phi_{\pm}^{-} - (\hbar z - 1) \alpha = 0, \quad (4.81)$$

and one introduces in addition,

$$\mathfrak{g} = -\frac{\phi_{+} + \phi_{-}}{2(\phi_{+} - \phi_{-})}, \quad (4.82)$$

$$\mathfrak{f} = \frac{1}{\phi_{+} - \phi_{-}}. \quad (4.83)$$

Using the Riccati-type equation (4.81) and its consequences,

$$\begin{aligned} \phi_{+} \phi_{+}^{-} - \phi_{-} \phi_{-}^{-} - (z + \beta)(\phi_{+}^{-} - \phi_{-}^{-}) &= 0, \\ \phi_{-}^{-} \phi_{+} \phi_{+}^{-} - \phi_{-} \phi_{-}^{-} \phi_{+}^{-} &= \alpha(\hbar z - 1)(\phi_{-}^{-} - \phi_{+}^{-}), \end{aligned}$$

one derives the identities

$$\mathbf{g} + \mathbf{g}^- + (z + \beta)\mathbf{f} = 0, \quad (4.84)$$

$$(\hbar z - 1)\alpha\mathbf{f}^- - (z + \beta)\mathbf{g}^- - (\hbar z - 1)\alpha^+\mathbf{f}^+ + (z + \beta)\mathbf{g} = 0, \quad (4.85)$$

$$\mathbf{g}^2 - \alpha^+(\hbar z - 1)\mathbf{f}\mathbf{f}^+ z = 1/4, \quad (4.86)$$

$$\mathbf{f} = \frac{1}{\phi_+ - \phi_-} = \frac{\phi_+^-\phi_-}{\alpha(\hbar z - 1)(\phi_- - \phi_+^+)} \quad (4.87)$$

Moreover, (4.84)-(4.87) also permit one to derive nonlinear difference equations for  $\mathbf{f}$  and  $\mathbf{g}$  separately, and one obtains

$$(z + \beta)^2 - (\mathbf{g} + \mathbf{g}^+)(\mathbf{g} + \mathbf{g}^-)(\hbar z - 1)\alpha^+ \quad (4.88)$$

$$= \frac{1}{4}(z + \beta)^2,$$

$$\begin{aligned} & ((z + \beta)^2\mathbf{f} + (\hbar z - 1)\alpha\mathbf{f}^- - (\hbar z - 1)\alpha^+\mathbf{f}^+)^2 \\ & - 4\mathbf{f}\mathbf{f}^+\alpha^+(\hbar z - 1)(z + \beta)^2 = (z + \beta)^2. \end{aligned} \quad (4.89)$$

**Theorem B.1** Assume (2.24),  $\text{s-RT}_p(\alpha, \beta) = 0$ , and suppose  $P = (z, y) \in \mathcal{K}_p \setminus \{P_{\infty\pm}\}$ . Then  $\mathbf{f}, \mathbf{g}$  has the following convergent expansions as  $|z| \rightarrow \infty$ ,

$$\begin{aligned} \mathbf{g} & \underset{|\zeta| \rightarrow 0}{=} \pm \sum_{\ell=0}^{\infty} \hat{g}_{\ell} \zeta^{\ell}, \quad \zeta = 1/z, \\ \mathbf{f} & \underset{|\zeta| \rightarrow 0}{=} \pm \sum_{\ell=0}^{\infty} \hat{f}_{\ell} \zeta^{\ell}, \quad \zeta = 1/z, \end{aligned} \quad (4.90)$$

and simultaneously as  $P \rightarrow P_{\infty\pm}$ ,

$$\begin{aligned} V_{11}/y &= \pm \sum_{\ell=0}^{\infty} \hat{g}_{\ell} \zeta^{\ell}, \quad \zeta \rightarrow 0, \quad \zeta = 1/z, \\ V_{12}/y &= \pm \sum_{\ell=0}^{\infty} \hat{f}_{\ell} \zeta^{\ell}, \quad \zeta \rightarrow 0, \quad \zeta = 1/z. \end{aligned} \quad (4.91)$$

Moreover, one infers for the  $E_m$ -dependent summation constants  $\delta_{\ell}, \ell = 0, \dots, p$ , in  $V_{ij}(i, j = 1, 2)$  that

$$\delta_{\ell} = c_{\ell}(\underline{E}) \quad \ell = 0, \dots, p \quad (4.92)$$

and

$$f_\ell = \sum_{k=0}^{\ell} c_{\ell-k}(\underline{E}) \hat{f}_k, \quad \ell = 0, \dots, p+1, \quad (4.93)$$

$$\begin{aligned} \hat{f}_\ell &= \sum_{k=0}^{\min\{\ell, p+1\}} \hat{c}_{\ell-k}(\underline{E}) f_k \\ &= \sum_{k=0}^{\min\{\ell, p\}} \hat{c}_{\ell-k}(\underline{E}) f_k, \quad \ell \in \mathbb{N}_0. \end{aligned} \quad (4.94)$$

**Proof.** Identifying

$$\Psi_+(z, \cdot) \text{ with } \Psi(P, \cdot, 0) \text{ and } \Psi_-(z, \cdot) \text{ with } \Psi(P^*, \cdot, 0), \quad (4.95)$$

and similarly, identifying

$$\phi_+(z, \cdot) \text{ with } \phi(P, \cdot) \text{ and } \phi_-(z, \cdot) \text{ with } \phi(P^*, \cdot), \quad (4.96)$$

a comparison of (4.80)-(4.89) and the result of Lemma 3.1 and 3.3 shows that we may also identify

$$\mathfrak{g} \text{ with } \pm \frac{V_{11}}{y}, \quad \mathfrak{f} \text{ with } \pm \frac{V_{12}}{y},$$

The sign depending on whether  $P$  tends to  $P_{\infty\pm}$ . Hence we are only to investigate the asymptotic expansions of  $V_{11}/y$  and  $V_{12}/y$ . Dividing  $V_{11}$  and  $V_{12}$  by  $y$ , one obtains

$$\frac{V_{11}(z)}{y} = \left( \sum_{k=0}^{\infty} \hat{c}_k(\underline{E}) z^{-k} \right) \left( \sum_{\ell=0}^{p+1} g_\ell z^{-\ell} \right) = \sum_{\ell=0}^{\infty} \check{g}_\ell z^{-\ell}, \quad (4.97)$$

$$\frac{V_{12}(z)}{y} = \left( \sum_{k=0}^{\infty} \hat{c}_k(\underline{E}) z^{-k} \right) \left( \sum_{\ell=0}^{p+1} f_\ell z^{-\ell} \right) = \sum_{\ell=0}^{\infty} \check{f}_\ell z^{-\ell}, \quad (4.98)$$

for some coefficients  $\check{g}_\ell, \check{f}_\ell$  to be determined next. Dividing (2.19) and (2.20) by  $y$  and inserting the expansions (4.97) and (4.98) into the resulting equation then yield the recursion relations (2.4)-(2.6) (with  $f_\ell$  replaced by  $\check{f}_\ell$ ). Moreover, plugging (4.97) and (4.98) into (2.32) and (2.33) then yields  $\check{g}_0 = -\frac{1}{2} = \hat{g}_0$ ,  $\check{g}_1 = \hbar\alpha^+ = \hat{g}_1$ ,  $\check{f}_0 = 0 = \hat{f}_0$ ,  $\check{f}_1 = 1 = \hat{f}_1$ . Then we can inductively to show that  $\check{g}_\ell = \mathfrak{M}_\ell(\hat{g}_0, \dots, \hat{g}_{\ell-1}) = \hat{g}_\ell$ ,  $\check{f}_\ell = \mathfrak{N}_\ell(\hat{f}_0, \dots, \hat{f}_{\ell-1}) = \hat{f}_\ell$ ,

where  $\mathfrak{M}_\ell, \mathfrak{N}_\ell$  are the polynomials in  $(\hat{g}_0, \dots, \hat{g}_{\ell-1})$  and  $(\hat{f}_0, \dots, \hat{f}_{\ell-1})$ , respectively. This implies (4.90) and (4.91). A comparison of coefficients in (4.98) then proves (4.94). Next, multiplying (3.45) and (3.46), a comparison of coefficients of  $\eta^{-k}$  yields

$$\sum_{\ell=0}^k \hat{c}_{k-\ell}(\underline{E}) c_\ell(\underline{E}) = \delta_{k,0} = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}. \quad (4.99)$$

Hence one computes

$$\begin{aligned} \sum_{k=0}^{\ell} c_{\ell-k}(\underline{E}) \hat{f}_k &= \sum_{m=0}^{\ell} c_{\ell-k}(\underline{E}) \left( \sum_{s=0}^{\min\{k, p+1\}} \hat{c}_{k-s}(\underline{E}) f_s \right) \\ &= \sum_{k=0}^{\ell} c_{\ell-k}(\underline{E}) \left( \sum_{s=0}^k \hat{c}_{k-s}(\underline{E}) f_s \right) \\ &= \sum_{k=0}^{\ell} \sum_{s=0}^k c_{\ell-k}(\underline{E}) \hat{c}_{k-s}(\underline{E}) f_s \\ &= \sum_{k=0}^{\ell} \sum_{s=0}^{\ell} c_{\ell-k}(\underline{E}) \hat{c}_{k-s}(\underline{E}) f_s \\ &= \sum_{s=0}^k \sum_{k=0}^{\ell} c_{\ell-k}(\underline{E}) \hat{c}_{k-s}(\underline{E}) f_s \\ &= \sum_{s=0}^{\ell} \left( \sum_{k=s}^{\ell} c_{\ell-k}(\underline{E}) \hat{c}_{k-s}(\underline{E}) \right) f_s \\ &= f_\ell, \quad \ell = 0, \dots, p+1. \end{aligned}$$

Hence one obtains (4.93) and thus (4.92).  $\square$

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